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Loop Quantum Gravity and Cosmology: A dynamical introduction

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Abstract

Loop quantum gravity and cosmology are reviewed with an emphasis on *evaluating* the dynamics, rather than constructing it. The three crucial parts of such an analysis are (i) deriving effective equations, (ii) controlling the theory's microscopic degrees of freedom that lead to its spatial discreteness and refinement, and (iii) ensuring consistency and anomaly-freedom. All three issues are crucial for making the theory testable by conceptual and observational means, and they remain challenging. Throughout this review, the Hamiltonian nature of the theory will play a large role for properties of space-time structure within the framework discussed.

"*It would be permissible to look upon the Hamiltonian form as the fundamental one, and there would then be no fundamental four-dimensional symmetry in the theory. One would have a Hamiltonian built up from four weekly [sic] vanishing functions, given by [the Hamiltonian and diffeomorphism constraints]. The usual requirement of four-dimensional symmetry in physical laws would then get replaced by the requirement that the functions have weakly vanishing P.b.'s, so that they can be provided with arbitrary coefficients in the equations of motion, corresponding to an arbitrary motion of the surface on which the state is defined.*"

P.A.M. Dirac (1958)

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1 Introduction

In its different incarnations, quantum gravity must face a diverse set of fascinating problems and difficulties, a set of issues best seen as both challenges and opportunities. One of the main problems in canonical approaches, for instance, is the issue of anomalies in the gauge algebra underlying space-time covariance. Classically, the gauge generators, given by constraints, have weakly vanishing Poisson brackets with one another: they vanish when the constraints are satisfied. After quantization, the same behavior must be realized for commutators of quantum constraints (or for Poisson brackets of effective constraints), or else the theory becomes inconsistent due to gauge anomalies. If and how canonical quantum gravity can be obtained in an anomaly-free way is an important question, not yet convincingly addressed in full generality. Posing one of the main obstacles to a complete formulation of quantum gravity, this issue is hindering progress toward a detailed evaluation of quantum gravitational dynamics. A reliable phenomenological analysis must, after all, start with a consistent set of sufficiently general dynamical equations.

But the strong and tough requirement of anomaly-freedom is also an opportunity, for it allows an analysis of quantum space-time and the changes in its structure possibly implied by quantum gravity. Addressing the anomaly problem is, moreover, crucial for an understanding of the dynamics of quantum gravity, both in the sense of *constructing* consistent dynamical equations at the quantum level and in the sense of *evaluating* equations and their solutions to bring out physical effects.

Although the anomaly problem has not been addressed in full generality, several model systems have by now been analyzed in loop quantum cosmology, as reviewed by Bojowald (2008c), with this question in mind. Loop quantum cosmology is a rather wide area within loop quantum gravity, analyzing several classes of model systems and perturbations around them. Loop quantum gravity, detailed by Rovelli (2004), Ashtekar and Lewandowski (2004) and Thiemann (2007), is a canonical quantization of general relativity based on holonomies (the eponymous loops) as elementary variables. The use of holonomies allows a background-independent formulation free of auxiliary metrics, and it implies several specific properties of the resulting dynamics.

In all the systems used in loop quantum cosmology, quantization techniques close to those of a general loop quantization are used; they can thus be seen as capturing at least some of the crucial properties of full loop quantum gravity. To different degrees, most of these models make additional use of symmetry reduction as introduced by Bojowald and Kastrup (2000), simplifying much of the quantum geometry and thereby providing rather direct access to the much less understood quantum dynamics. Thanks to these steps, implications of the quantum dynamics of loop quantum gravity, as generally defined based on Rovelli and Smolin (1994) and Thiemann (1998a), have been evaluated quite explicitly for the first time.

In general terms as well as for particular questions arising in loop quantum cosmology and loop quantum gravity, three key issues regarding quantum space-time arise, not unlike what one would expect for any fundamental gauge theory with microscopic degrees of freedom:

Effective dynamics: In quantum gravity, geometry is described unsharply by whole states with all their fluctuations and correlations, in addition to the expectation values for an average geometry. Equations of motion for expectation values receive quantum corrections in their effective dynamics, as it may describe a quantum geometry. Along with this consequence of quantizing gravity come not only new mathematical space-time structures but also a vast enlargement of the number of degrees of freedom by quantum variables.

The strongest control of such a high-dimensional dynamical quantum system is usually obtained for dynamical coherent states, defined as states saturating uncertainty relations at all times. If such states exist, they provide insights into the minimal deviations from classical behavior expected for a quantum system. The form and behavior of dynamical coherent states in loop quantum cosmology can be highlighted in several models, bringing out the role of space-time fluctuations and correlations as degrees of freedom beyond the classical ones. Exact dynamical coherent states exist only in special models and for specific initial values. Nevertheless, they allow interesting views on the generic quantum dynamics as it arises in quantum gravity.

Before all corrections are derived for a large class of models, a clear analysis provided by dynamical coherent states, when they exist, unambiguously shows the first deviations from classical behavior. More generally, when exact dynamical coherent states do not exist, additional quantum corrections will result. They can often be computed perturbatively, analogously to loop corrections in interacting quantum field theories.

Discrete dynamics: In addition to those generic effects due to non-classical state parameters, underlying space-time structures typically change even for the expectation values of a quantum-gravity state. Most importantly, discrete geometry, at least in purely spatial terms which is by now well understood in loop quantum gravity following Rovelli and Smolin (1995); Ashtekar and Lewandowski (1997, 1998), shows several detailed properties of importance for the dynamics and thus for space-time geometry.

A spatial slice in space-time is equipped with a discrete quantum geometry, roughly seen as making space built from atomic patches of certain discrete sizes. One of the main problems of quantum dynamics is to show how these spatial atoms along slices fit together to form a quantum space-time — or, more dynamically, how the spatial atoms change, merge, subdivide and interact as time is let loose. For an expanding universe, one would expect the discrete spatial structure to be refined as the volume increases; otherwise discrete sizes would be enlarged by huge factors, especially during inflation, making them macroscopic. The full dynamics of loop quantum gravity has indeed provided several hints that the number of discrete building blocks must change from slice to slice, once the dynamics is implemented consistently. This lattice refinement can be modelled even in the simplest, most highly symmetric situations of loop quantum cosmology, laid out by Bojowald (2006, 2008a). And it has shown several specific implications by which its precise form can already be constrained.

Consistent dynamics: Dynamics unfolds in time, but time is relative. Making sure that descriptions using different notions of time, corresponding to measurements by different observers, can agree about their physical insights requires the consistency conditions of general covariance. Since general covariance in gravity is implemented by gauge transformations, any quantization or even just a modification of the theory must, for consistency, respect this principle and be anomaly free.

While the previous two points manifest themselves already in homogeneous models, where they can most easily be studied, the anomaly problem arises only in inhomogeneous situations. Within homogeneity, all but one of the gauge transformations underlying covariance are fixed. Anomalies can only arise if there are at least two independent gauge transformations; after quantization, they would be anomalous if their composition is no longer a gauge transformation. One could, of course, make it a gauge transformation by definition, by declaring the whole group generated by the independent gauge transformations as the gauge group. But if this group is too large, it would identify variables which are to be considered physically distinct, removing observables and degrees of freedom and in many cases leaving no non-trivial solutions. Changing the gauge transformations of a classical theory by quantum effects requires much care; only so-called consistent deformations of the classical gauge generators can provide well-defined quantizations.

Addressing these questions is crucial, not only for a complete formulation of quantum gravity but also for reliable cosmological applications based on the resulting set of equations (such as singularity removal or structure formation). In what follows, we will describe the current status based on the models of loop quantum cosmology.

2 Effective dynamics

In a general sense, effective equations of a quantum system describe the behavior of expectation values in a state. Deriving such equations for the expectation values of basic operators, such as $\hat{q}$ and $\hat{p}$ in quantum mechanics, shows how quantum effects change the classical equations of motion. If the equations can be solved or analyzed, the manifestation of quantum behavior will be seen. Such effects play an important role in any interacting quantum theory, so also in quantum gravity and especially in quantum cosmology which devotes itself to the analysis of extremely long evolution times. During those times, quantum states may change drastically, and quantum corrections grow to be significant.

2.1 Momentous quantum mechanics

Effective equations are state dependent since they are equations for expectation values in a state with many more independent parameters. In general, quantum fluctuations will influence the behavior of expectation values, and must then be included in effective equations in some form. More generally, and following Bojowald and Skirzewski (2006),

we can parameterize a state by all its moments

$$G^{a,b} \equiv G^{\overbrace{q\cdots q}^{a}\overbrace{p\cdots p}^{b}} = \langle(\hat{q}-\langle\hat{q}\rangle)^a(\hat{p}-\langle\hat{p}\rangle)^b\rangle_{\rm Weyl} \tag{1}$$

defined for $a+b \geq 2$, in addition to the expectation values $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$. (The subscript "Weyl" denotes totally symmetric ordering. We will use the two notations indicated on the left interchangeably, at least for small $a+b$.) For instance, $G^{2,0} \equiv G^{qq}$ is the square of position fluctuations, and $G^{1,1} \equiv G^{qp} = \frac{1}{2}\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle - \langle\hat{q}\rangle\langle\hat{p}\rangle$ the covariance. The values of moments are not completely arbitrary, most importantly being restricted by uncertainty relations such as

$$G^{qq}G^{pp} - (G^{qp})^2 \geq \frac{\hbar^2}{4}\,. \tag{2}$$

For pure states, the set of moments as defined here is overcomplete; the framework more generally allows for mixed states, too.

All the moments are dynamical. Given a Hamiltonian operator, for every observable $\hat{O}$ we have an equation of motion

$$\frac{{\rm d}\langle\hat{O}\rangle}{{\rm d}t} = \frac{\langle[\hat{O},\hat{H}]\rangle}{i\hbar} \tag{3}$$

of its expectation value. Specific examples are obtained for the terms in a moment, and so we can derive their equations of motion. For the square of position fluctuations, for instance, we have

$$\frac{{\rm d}G^{qq}}{{\rm d}t} = \frac{\rm d}{{\rm d}t}(\langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2) = \frac{\langle[\hat{q}^2,\hat{H}]\rangle}{i\hbar} - 2\langle\hat{q}\rangle\frac{\langle[\hat{q},\hat{H}]\rangle}{i\hbar}\,.$$

Introducing Poisson brackets on the space of moments via

$$\{\langle\hat{A}\rangle,\langle\hat{B}\rangle\} = \frac{\langle[\hat{A},\hat{B}]\rangle}{i\hbar}\,, \tag{4}$$

extended by linearity and the Leibniz rule, equations of motion take Hamiltonian form:

$$\frac{{\rm d}G^{a,b}}{{\rm d}t} = \{G^{a,b}, H_Q\} \tag{5}$$

with the quantum Hamiltonian $H_Q := \langle\hat{H}\rangle$.

In general, the moments are all coupled to one another and to the equations for expectation values. One can see this by expanding

$$\begin{aligned} H_Q(\langle\hat{q}\rangle,\langle\hat{p}\rangle,G^{a,b}) &= \langle H(\hat{q},\hat{p})\rangle = \langle H(\langle\hat{q}\rangle + (\hat{q}-\langle\hat{q}\rangle), \langle\hat{p}\rangle + (\hat{p}-\langle\hat{p}\rangle))\rangle \\ &= H(\langle\hat{q}\rangle,\langle\hat{p}\rangle) + \sum_{a,b:a+b\geq 2} \frac{1}{a!b!}\frac{\partial^{a+b}H(\langle\hat{q}\rangle,\langle\hat{p}\rangle)}{\partial\langle\hat{q}\rangle^a\partial\langle\hat{p}\rangle^b}G^{a,b} \end{aligned} \tag{6}$$

(where we assumed the Hamiltonian operator $\hat{H}$ to be Weyl-ordered in $\hat{q}$ and $\hat{p}$) and noticing the coupling terms of expectation values and moments for any non-quadratic

potential. The Hamiltonian flow (3) or (5) then couples expectation values $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ to all the moments.

At this stage, we have an exact but usually horribly complicated Hamiltonian description of quantum evolution. The partial differential equation for a state in Schrödinger's formulation is replaced by infinitely many ordinary differential equations for the moments. Most of the labor that goes into deriving tractable effective equations consists in extracting the required information about expectation values without having to solve for a full state, or all its moments, and to specify the regimes where this is reliable. In a semiclassical approximation based on near-Gaussian states, for instance, a moment of order $a+b$ is typically of the order $\hbar^{(a+b)/2}$, giving rise to a natural expansion in powers of $\hbar$. To any given order, only finitely many moments need be considered.[1]

2.2 Harmonic oscillator

In special systems, equations of motion for the moments decouple automatically to finite sets. The best known case is the Harmonic oscillator, whose quantum Hamiltonian is

$$H_Q = \frac{1}{2m}\langle\hat{p}\rangle^2 + \frac{1}{2}m\omega^2\langle\hat{q}\rangle^2 + \frac{1}{2}m\omega^2 G^{qq} + \frac{1}{2m}G^{pp}\,. \tag{7}$$

Second-order moments appear, but they do not couple to the expectation values. They rather provide the zero-point energy due to quantum fluctuations. Hamiltonian equations of motion are only finitely coupled,

$$\begin{aligned}
\frac{\mathrm{d}\langle\hat{q}\rangle}{\mathrm{d}t} &= \{\langle\hat{q}\rangle, H_Q\} = \frac{1}{m}\langle\hat{p}\rangle \\
\frac{\mathrm{d}\langle\hat{p}\rangle}{\mathrm{d}t} &= \{\langle\hat{p}\rangle, H_Q\} = -m\omega^2\langle\hat{q}\rangle \\
\frac{\mathrm{d}G^{a,b}}{\mathrm{d}t} &= \{G^{a,b}, H_Q\} = \frac{1}{m}aG^{a-1,b+1} - m\omega^2 bG^{a+1,b-1}\,.
\end{aligned} \tag{8}$$

For the expectation values we have exactly the classical equations without quantum corrections, as is well-known for the harmonic oscillator. Here, effective equations for $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ are identical to the classical ones.

The remaining equations then show how the state evolves once initial values for moments have been chosen. For stationary states, for instance, a vanishing covariance G^{qp} ensures that fluctuations are time-independent. The covariance is constant in time if $\dot{G}^{qp} = G^{pp}/m - m\omega^2 G^{qq} = 0$, or $G^{pp} = m^2\omega^2 G^{qq}$. With this, the non-classical contribution to the energy H_Q in (7) is $m\omega^2 G^{qq}$, where $G^{qq} \geq \hbar/2m\omega$ from (2). A minimal two-point

[1] Truncating the phase space in this way leads to Poisson manifolds spanned by the moments to a certain order. In general, these Poisson structures are degenerate; for instance, there are three independent second order moments, forming a space which cannot carry a non-degenerate Poisson structure. Effective equations thus make crucial use of Poisson geometry, not symplectic geometry as in other geometric formulations of quantum mechanics such as the one going back to Kibble (1979).

energy $\frac{1}{2}\hbar\omega$ results for the ground state, derived purely by effective means (even though the ground state is not at all semiclassical).

Decoupled equations of this form are very useful because they can directly show important aspects of quantum dynamics. Expectation-value equations decouple from the rest whenever the algebra of basic operators together with the Hamiltonian is linear: in this case, the time derivative of the expectation value of any one of the basic operators is an expectation value of a basic operator. Such systems are solvable in a strong sense; there is no quantum back-reaction from the moments on the dynamics of expectation values. And the dynamics of moments of a given order depends only on moments of the same order. In terms of quantum field theory, solvable models correspond to free theories.

2.3 Low-energy effective potential

With interactions, or for anharmonic terms, quantum back-reaction results. Moments couple non-trivially to expectation values and become important for their dynamics. In this way, the state dependence of effective equations ensues. The system of equations necessarily becomes of higher dimension than classically, with new dynamical quantum degrees of freedom given by the moments.

For a quantum mechanical system with an anharmonic potential, for instance, we have the classical Hamiltonian $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 + U(q)$ with anharmonicity $U(q)$. In terms of dimensionless quantum variables

$$g^{a,b} = \hbar^{-(a+b)/2}(m\omega)^{a/2-b/2}G^{a,b} \tag{9}$$

the quantum Hamiltonian is

$$H_Q = \frac{1}{2m}\langle\hat{p}\rangle^2 + \frac{1}{2}m\omega^2\langle\hat{q}\rangle^2 + U(\langle\hat{q}\rangle) + \frac{\hbar\omega}{2}(g^{2,0}+g^{0,2}) + \sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{\hbar}{m\omega}\right)^{n/2} U^{(n)}(\langle\hat{q}\rangle)g^{n,0} \tag{10}$$

and generates equations of motion

$$\begin{aligned}
\frac{\mathrm{d}\langle\hat{q}\rangle}{\mathrm{d}t} &= \frac{1}{m}\langle\hat{p}\rangle \\
\frac{\mathrm{d}\langle\hat{p}\rangle}{\mathrm{d}t} &= -m\omega^2\langle\hat{q}\rangle - U'(\langle\hat{q}\rangle) - \sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{\hbar}{m\omega}\right)^{n/2} U^{(n+1)}(\langle\hat{q}\rangle)g^{n,0} \\
\dot{g}^{a,b} &= a\omega g^{a-1,b+1} - b\omega g^{a+1,b-1} - \frac{bU''(\langle\hat{q}\rangle)}{m\omega}g^{a+1,b-1} \\
&\quad + \frac{b\sqrt{\hbar}U'''(\langle\hat{q}\rangle)}{2(m\omega)^{3/2}}\left(g^{a,b-1}g^{qq} - g^{a+2,b-1} + \frac{(b-1)(b-2)}{12}g^{a,b-3}\right) \\
&\quad + \frac{b\hbar U''''(\langle\hat{q}\rangle)}{6(m\omega)^2}\left(g^{a,b-1}g^{qqq} - g^{a+3,b-1} + \frac{(b-1)(b-2)}{4}g^{a+1,b-3}\right) + \cdots .
\end{aligned} \tag{11}$$

The leading quantum correction appears in the equation for $\langle\hat{p}\rangle$ at order $\hbar$ ($n=2$), for which we have to know the position fluctuation $g^{2,0} = g^{qq}$. In general, this is an

independent variable subject to its own equation of motion. Its evolution couples to other quantum variables, eventually making the whole infinite system coupled. At this stage, approximations are required. For semiclassical states, we may drop terms of higher order in $\hbar$,[2] providing a closed system of effective equations

$$
\begin{aligned}
\frac{\mathrm{d}\langle\hat{q}\rangle}{\mathrm{d}t} &= \frac{1}{m}\langle\hat{p}\rangle \\
\frac{\mathrm{d}\langle\hat{p}\rangle}{\mathrm{d}t} &= -m\omega^2\langle\hat{q}\rangle - U'(\langle\hat{q}\rangle) - \frac{1}{2}\frac{\hbar}{m\omega}U'''(\langle\hat{q}\rangle)g^{qq} + O(\hbar^{3/2}) \\
\frac{\mathrm{d}g^{qq}}{\mathrm{d}t} &= 2\omega g^{qp} \\
\frac{\mathrm{d}g^{qp}}{\mathrm{d}t} &= \omega(g^{pp} - g^{qq}) - \frac{U''(\langle\hat{q}\rangle)}{m\omega}g^{qq} + O(\sqrt{\hbar}) \\
\frac{\mathrm{d}g^{pp}}{\mathrm{d}t} &= -2\omega g^{qp} - 2\frac{U''(\langle\hat{q}\rangle)}{m\omega}g^{qp} + O(\sqrt{\hbar})
\end{aligned}
$$

for expectation values and second-order moments. With higher moments dropped, this system shows the leading quantum corrections in a finitely-coupled system.

For anharmonic oscillators, a further treatment is possible which makes use of an adiabatic approximation: we assume that time derivatives of the moments are small compared to the other terms. In this case, equations of motion for moments become algebraic relationships between them. For the leading adiabatic order $g_0^{a,b}$, combined with the previous $\hbar$-expansion, we must solve

$$
0 = \{g_0^{a,b}, H_Q\} = \omega\left(a g_0^{a-1,b+1} - b\left(1 + \frac{U''(\langle\hat{q}\rangle)}{m\omega^2}\right) g_0^{a+1,b-1}\right) + O(\sqrt{\hbar})
$$

which is of the stationary harmonic form, but with position-dependent coefficients. Although $\mathrm{d}\langle\hat{q}\rangle/\mathrm{d}t \neq 0$, this is consistent with the adiabatic assumption. The general solution is

$$
g_0^{a,b} = \binom{(a+b)/2}{b/2}\binom{a+b}{b}^{-1}\left(1 + \frac{U''(\langle\hat{q}\rangle)}{m\omega^2}\right)^{b/2} g_0^{a+b,0}
$$

for even a and b, and $g_0^{a,b} = 0$ whenever a or b are odd. The values of $g_0^{n,0}$ for n even remain free, but must satisfy a condition following from the next adiabatic order $g_1^{a,b}$.

For the first adiabatic order, we now consider $g_0^{a,b}$ weakly time-dependent via $\langle\hat{q}\rangle$, but assume $g^{a,b} - g_0^{a,b}$ time independent. Solutions to the resulting equations $\{g_1^{a,b} - g_0^{a,b}, H_Q\} = 0$ provide the first adiabatic order[3] $g_1^{a,b}$, whose time dependence is given by derivatives of

[2] By definition (9), the leading semiclassical $\hbar$-dependence of $g^{a,b}$ has been factored out, such that explicit factors of $\hbar$ in the equations of motion suffice to read off orders. In terms of the original moments, at second order we ignore all terms $\hbar^n G^{a,b}$ with $2n + a + b > 2$.

[3] The n-th adiabatic order $g_n^{a,b}$ is assumed to satisfy $\{g_n^{a,b} - g_{n-1}^{a,b}, H_Q\} = 0$ and thus solves $\{g_n^{a,b}, H_Q\} = \dot{g}_{n-1}^{a,b}$. Iterating over n, this provides algebraic equations for all orders.

the zeroth (adiabatic) order moments:

$$\{g_1^{a,b}, H_Q\} = \omega\left(a g_1^{a-1,b+1} - b\left(1 + \frac{U''(\langle\hat{q}\rangle)}{m\omega^2}\right) g_1^{a+1,b-1}\right) = \dot{g}_0^{a,b}.$$

This equation implies

$$\sum_{b\text{ even}} \binom{(a+b)/2}{b/2}\left(1 + \frac{U''(\langle\hat{q}\rangle)}{m\omega^2}\right)^{b/2} \dot{g}_0^{a,b} = 0$$

and requires $g_0^{n,0} = C_n(1 + U''(\langle\hat{q}\rangle)/m\omega^2)^{-n/4}$. The remaining constant C_n can be fixed by requiring the moments to be of harmonic-oscillator form for $U = 0$: $C_n = 2^{-n} n!/(n/2)!$. In particular, $g_0^{qq} = \frac{1}{2}(1 + U''(\langle\hat{q}\rangle)/m\omega^2)^{-1/2}$, and the quantum correction to the effective force is $-\frac{1}{4}(\hbar/m\omega)U'''(\langle\hat{q}\rangle)(1 + U''(\langle\hat{q}\rangle)/m\omega^2)^{-1/2}$ as it arises from an effective potential

$$V_{\text{eff}}(q) = \frac{1}{2}m\omega^2 q^2 + U(q) + \frac{1}{2}\hbar\omega\sqrt{1 + \frac{U''(q)}{m\omega^2}} \tag{12}$$

from $\frac{1}{2}\hbar\omega(g_0^{qq} + g_0^{pp}) + \frac{1}{2}(\hbar/m\omega)U''(\langle\hat{q}\rangle)g_0^{qq}$ in (10). This function agrees with path-integral calculations of the low-energy effective action as derived by Cametti et al. (2000). Here, only the zeroth adiabatic order combined with first order in $\hbar$ has been used. To second order computed by Bojowald and Skirzewski (2006), there is also a correction for the mass term, still in agreement with Cametti et al. (2000).

The adiabatic approximation is an example for situations in which the coupled quantum evolution of moments can be further reduced to result in explicit effective forces. While the forces come from coupling terms between expectation values and quantum variables such as fluctuations, the latter do not appear explicitly. Quantum effects then manifest themselves indirectly in the form of effective terms depending on the expectation values, their quantum origin being indicated only by the presence of $\hbar$ but not by explicit quantum degrees of freedom. Such effective terms cannot always be derived since some regimes, where effective descriptions may well apply, do require the larger freedom of higher-dimensional effective systems coupling quantum degrees of freedom explicitly. For low-energy effective potentials, the adiabatic approximation is responsible for the reduction to a system of classical form as far as degrees of freedom are concerned.

From the derivation we can also see how the state dependence is a crucial part of effective equations, even though the final expression (12) for the potential seems state independent. To fix all free constants in effective equations for anharmonic systems, in particular C_2, we had to refer to the harmonic-oscillator ground state. We are thus expanding around the vacuum of the free, solvable theory whose moments are known. Quantum corrections from the interacting vacuum have been derived in passing, by obtaining the leading adiabatic orders of g^{qq}. For the low-energy effective potential, this result was re-inserted in the equations of motion, somewhat hiding the state dependence.

With the higher-dimensional viewpoint of effective systems, keeping some of the moments as independent parameters, we obtain extra information about the interacting theory

not seen in a simple effective potential. For instance, from our intermediate calculations we directly have $g_0^{qq} = \frac{1}{2}(1 + U''(\langle\hat{q}\rangle)/m\omega^2)^{-1/2}$, while $g_0^{pp} = (1 + U''(\langle\hat{q}\rangle)/m\omega^2)g_0^{qq} = \frac{1}{2}(1 + U''(\langle\hat{q}\rangle)/m\omega^2)^{1/2}$ and $g_0^{qp} = 0$. To zeroth adiabatic order, the interacting ground state keeps saturating the uncertainty relation, but fluctuations are no longer exactly constant.

2.4 Quantum cosmology

In relativistic systems, there is no absolute time with evolution generated by a Hamiltonian. Rather, relativistic systems are subject to a Hamiltonian constraint C. It generates arbitrary changes of the time coordinate as gauge transformations $\delta_\epsilon f = \epsilon\{f, C\}$ for phase-space functions f. From observable quantities O left unchanged by gauge transformations, that is $\{O, C\} = 0$, dynamical properties follow. Since the invariance condition $\{O, C\} = 0$ removes one dimension from the initial phase space, for consistency we must require $C = 0$ as a constraint.[4]

Constrained formulations can be introduced also for non-relativistic systems by parameterization, adding a time degree of freedom t with momentum p_t and replacing the Hamiltonian H by the Hamiltonian constraint $C = p_t - H$. For time-independent Hamiltonians, p_t is gauge invariant while the gauge transformation of t is $\delta_\epsilon t = \epsilon\{t, C\} = \epsilon$; t can be changed at will. For the remaining observables,

$$0 = \{O, C\} = \frac{\mathrm{d}O}{\mathrm{d}t} - \{O, H\}$$

imposes Hamilton's equations of motion.

For relativistic quantum systems, the effective techniques described so far cannot directly be applied. There are extensions to effective constraints briefly described later, as developed by Bojowald et al. (2009a); Bojowald and Tsobanjan (2009). But more simply, effective equation techniques can be used if systems are first deparameterized, reverting the above procedure. If there is a variable ϕ, then called internal time, such that the Hamiltonian constraint can be written as $C = p_\phi - H(q, p)$ with the momentum p_ϕ of ϕ and a function H independent of ϕ and p_ϕ, gauge transformations generated by the constraint take the form

$$\delta_\epsilon f = \epsilon\{f(q, p, \phi, p_\phi), C\} = \epsilon\left(\frac{\partial f}{\partial\phi} - \{f, H\}\right).$$

[4] Factoring out the Hamiltonian flow generated by the constraint C via its Hamiltonian vector field $X_C = \{\cdot, C\}$, we obtain a projection $\pi\colon M \to M/X_C$ from the original phase space M by identifying all points along the orbits of X_C. All observables O naturally descend to the factor space since they are constant along the orbits, and so does C. In this way, we obtain a complete set of functions on M/X_C. On the factor space, we have a natural Poisson structure $\{f, g\}_{M/X_C} = \{\pi^* f, \pi^* g\}_M$, pushing forward the Poisson bivector via π. This Poisson structure is degenerate: $\{O, C\}_{M/X_C} = 0$ for all functions O on M/X_C. The constraint C becomes a Casimir function on the factor space, and symplectic leaves of the Poisson structure are given by $C = \text{const}$. Any leaf carries a non-degenerate symplectic structure and can be taken as a reduced phase space, but $C = 0$ is distinguished: In this case, we can write gauge transformations as $\delta_\epsilon f = \{f, \epsilon C\}$ even for phase-space functions ϵ. More details of Poisson geometry in the context of constrained systems are described by Bojowald and Strobl (2003).

Gauge invariant quantities of the theory are thus those evolving in the usual Hamiltonian way as generated by the ϕ-Hamiltonian H.

Once deparameterized, the observables of a constrained system can be derived by analyzing an ordinary Hamiltonian flow. At this stage, effective techniques as described before can be applied to quantizations of deparameterized models. Effective equations of motion derived from

$$\frac{\mathrm{d}\langle\hat{O}\rangle}{\mathrm{d}\phi} = \frac{\langle[\hat{O},\hat{H}]\rangle}{i\hbar} \tag{13}$$

then provide means to solve for quantum observables $\langle\hat{O}\rangle(\phi)$ and their physical evolution. For the initial constrained system, solutions $\langle\hat{O}\rangle(\phi)$ are observables as functions on the full phase space including (ϕ, p_ϕ). (Remaining phase space variables enter the expression $\langle\hat{O}\rangle(\phi)$ via initial values taken for them when solving the differential equations (13).) The deperameterization endows $\langle\hat{O}\rangle(\phi)$ with the interpretation as an observable $\langle\hat{O}\rangle$ evolving with respect to the internal time ϕ. This is the relational picture for interpreting constrained dynamics, developed classically by Bergmann (1961), Rovelli (1991b,a) and Dittrich (2007, 2006).

Examples for such systems in cosmology are homogeneous models sourced by a free, massless scalar. Its energy density is purely kinetic, $\rho = \frac{1}{2}a^{-6}p_\phi^2$, such that the Friedmann equation, solved for p_ϕ, provides a Hamiltonian for ϕ-evolution. For a spatially flat Friedmann–Robertson–Walker model, the ϕ-Hamiltonian then turns out to be quadratic in suitable canonical variables: for instance, the Hubble parameter $\mathcal{H} = \dot{a}/a$ is canonically conjugate to the volume $V = a^3$, and the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{4\pi G}{3}\frac{p_\phi^2}{a^6}$$

tells us that $p_\phi^2 \propto V^2\mathcal{H}^2$. Upon taking a square root, we have a quadratic ϕ-Hamiltonian. Based on this observation, state properties have been determined by Bojowald (2007b,a). (Strictly speaking, the ϕ-Hamiltonian is of the form $|qp|$ which is not quadratic. However, for effective equations one can show that the absolute value can be dropped, providing a linear quantum system. We only have to require an initial state to be supported on a definite part of the spectrum of $\widehat{qp}$, either the positive or the negative one, which is then preserved in time since $\hat{H}$ is preserved. The absolute value just amounts to multiplication with ± 1. For initially semiclassical states, this requirement does not lead to restrictions for low-order moments.)

For other systems, perturbation theory can be used as described above for anharmonic oscillators. A crucial difference, according to the analysis by Bojowald et al. (2007a), is that no adiabatic regime has so far been found for quantum cosmology, blocking the complete expression of quantum variables in terms of an effective potential. On the other hand, higher-dimensional effective systems, where quantum variables are taken as independent variables subject to their own evolution, can be analyzed and show how states back-react on the expectation-value trajectories.

If we include a mass term or a potential for the scalar, the system becomes "time-dependent" in ϕ. Extra care is required, but perturbation theory still applies for small and flat potentials. The justification for this procedure in the time-dependent case comes from an extension of the effective-equation procedure to constrained systems introduced by Bojowald et al. (2009a); Bojowald and Tsobanjan (2009) without requiring deparameterization. Quantum constraint operators then imply the existence of infinitely many constraints $\langle\widehat{\mathrm{pol}}\hat{C}\rangle$ on the quantum phase space, in general all independent for different polynomials $\widehat{\mathrm{pol}}$ in the basic operators. This large number of constraints restricts not only expectation values (as the classical analog) but also the corresponding quantum variables. A complete reduction to the physical state space results, without requiring a deparameterization to exist. With these general techniques, effective equations for relativistic systems are thus fully justified. As discussed in Sec. 4, inhomogeneous models bring in a new level due to the anomaly problem, which does not arise for a single classical constraint.

2.5 Symplectic structure

Quantum corrections in canonical effective equations come from corrected Hamiltonians or corrected constraints as expectation values of Hamiltonian (constraint) operators. The Hamiltonian H or constraints C_I, together with Lagrange multipliers N^I (for gravity lapse and shift multiplying the Hamiltonian and diffeomorphism constraints), form an important contribution to the action. But the action

$$S[q(t),p(t);N^I] = \int \mathrm{d}t \left(\dot{q}p - H(q,p) - N^I C_I(q,p)\right)$$

of a Hamiltonian system in canonical form has an extra contribution, the one that determines the symplectic structure between configuration and momentum variables. One might wonder whether it is enough to look for quantum corrections in the Hamiltonian or the constraints without correcting the symplectic structure. With symplectic-structure corrections, an effective action might take a different form than suggested by an analysis only of Hamiltonians and constraints.

There might indeed be corrections to the symplectic structure, but they would follow from the same algebraic notions used for canonical effective equations. Poisson brackets of quantum variables are given by expectation values of commutators as used before; see (4). Any potential corrections to the symplectic structure have thus been taken care of by applying that formula consistently. If there are no changes in Poisson-bracket relations for effective equations, it is only because they do not change for expectation values of basic operators: they have commutator relations mimicking the classical Poisson algebra, which is linear for the basic objects. Taking expectation values of these linear structures does not lead to corrections. For canonical basic operators $\hat{q}$ and $\hat{p}$, for instance, we have $\{\langle\hat{q}\rangle,\langle\hat{p}\rangle\} = \langle[\hat{q},\hat{p}]\rangle/i\hbar = 1$.

The Poisson structure changes only because the dimension of the phase space increases by the quantum variables $G^{a,b}$. These variables satisfy Poisson relationships following from the quantum theory, but this does not affect the Poisson brackets of expectation values of

basic operators. In an action, the symplectic term for classical variables remains unchanged. But if a higher-dimensional effective system with independent quantum variables is used, their symplectic terms add to the action. The symplectic structure is only extended to include new degrees of freedom; it is not quantum corrected.

Sometimes, one can make assumptions[5] about the dependence of quantum variables on expectation values, or even derive those by an adiabatic approximation. If this is done, one can insert such expressions, schematically $G^I(q,p)$, into the symplectic form for quantum variables, of the form $\Omega_{IJ}(G)\mathrm{d}G^I\wedge\mathrm{d}G^J$ where $\Omega_{IJ}(G)$ follows from (4) purely kinematically. Then, a term $2\Omega_{IJ}(\partial G^I/\partial q)(\partial G^J/\partial p)\mathrm{d}q\wedge\mathrm{d}p$ results which would add to the symplectic term of expectation values. Now, the correction has dynamical information via the partial solutions $G^I(q,p)$. (For the specific example of anharmonic oscillators, no such corrections arise since the effective equations for g^{ab}, and thus their solutions, depend only on $\langle\hat{q}\rangle$, not $\langle\hat{p}\rangle$.) With independent quantum variables, as they are most often required, no symplectic-structure corrections result. Expectation values of Hamiltonians or constraints are then the key source for quantum corrections.

For quantum gravity, we expect higher-curvature terms in an effective action, and thus higher-derivative terms. By the preceding discussion, this can only come from independent moments $G^{a,b}$. But quantum gravity has additional implications, such as the emergence of discrete spatial structures expected in particular from loop quantum gravity. They, too, must affect the effective dynamics.

3 Discrete dynamics

Our main interest from now on will be to apply effective equation techniques to canonical quantum gravity. A strict derivation requires detailed knowledge of the mathematical properties of operators involved, as well as information about the representation of states. Effective equations and effective constraints are, after all, obtained via expectation values of Hamilonians or Hamiltonian constraint operators. These operators are first constructed within the full quantum theory of gravity or one of its models, and the resulting constructs will be sensitive to properties of the basic operators, analogous to $\hat{q}$ and $\hat{p}$ in quantum mechanics. Some of these properties will descend to the effective level once expectation values of Hamiltonians are computed, and then affect also the effective dynamics. At this stage, general effective techniques naturally tie in with specific constructions of a concrete quantum theory at hand.

3.1 Loop quantum gravity

Classically, the canonical structure of general relativity is given by the spatial part q_{ab} of the space-time metric g_{ab}, as well as momenta related to its change in time, or the extrinsic curvature K_{ab} of the spatial slices Σ. These specific quantities are meaningful

[5]A quantum cosmological model with assumptions for semiclassical states has been analyzed by Taveras (2008).

only when a choice for time, a time function t such that $\Sigma\colon t = \text{const}$, has been made, and so this formalism is often seen as breaking covariance. But space-time covariance is broken only superficially and is restored when all the dynamical constraints have been solved — in addition to a local Hamiltonian constraint, three components of the diffeomorphism constraint to generate all four independent space-time coordinate changes. The theory is, after all, equivalent to general relativity in its Lagrangian form; just formulating it in different variables cannot destroy underlying symmetries. In fact, the Hamiltonian formulation still shows the full generators of all gauge transformations in explicit form, by the constraints it implies for the fields. By the general theory of constraint analysis, a discussion of gauge at the Hamiltonian level[6] is then even more powerful than at the Lagrangian one.

3.1.1 Smearing

Before addressing quantum constraints, the quantum theory must be set up. For a well-defined quantization one turns the basic fields into operators such that the classical Poisson bracket is reflected in commutator relationships. In constructions of quantum field theories one has to face the problem that Poisson brackets of the fields involve delta functions since they are non-vanishing only when the values of two conjugate fields are taken at the same point, as in $\{\phi(x), p_\phi(y)\} = \delta(x,y)$ for a scalar field. A simple but powerful remedy is to "smear" the fields by integrating them against test functions over space. Again for a scalar field, we could use $\phi[\mu] := \int \mathrm{d}^3x \sqrt{\det q}\,\mu(x)\phi(x)$ for which $\{\phi[\mu], p_\phi(y)\} = \int \mathrm{d}^3x \mu(x)\delta(x,y) = \mu(y)$. The Poisson algebra for the smeared fields is free of delta functions thanks to the integration. Moreover, for sufficiently general classes of test functions $\mu(x)$, smeared fields capture the full information contained in the local fields and can be used to set up a general theory.

After smearing, a well-defined Poisson algebra of basic objects results, ready to be turned into an operator algebra by investigating its representations. For gravity, we would use smeared versions of the spatial metric and its change in time. But here, a second problem arises. One of the dynamical fields to be quantized is the spatial metric, but to smear fields we need a metric for the integration measure. This is no problem when fields to be quantized are non-gravitational. As with the scalar in the example above, we would use the background space-time on which the scalar moves, obtaining quantum field theory on a given, possibly curved background. But what do we do if the metric itself is one of the fields to be quantized? If we use it to smear itself, the resulting object becomes ugly, non-linear and too complicated as one of the basic quantities of a quantum field theory. If we introduce a separate metric just for the purpose of smearing the dynamical fields of gravity, this auxiliary input is likely to remain in the results derived from the theory. We would be quantizing gravitational excitations on a given space-time background, not the full gravitational field or space-time itself. We would be violating the great insight of general relativity by formulating physics on an auxiliary space-time, rather than realizing

[6]For an introduction to these techniques see Bojowald (2010).

gravity as the manifestation of space-time geometry.

Fortunately, it is possible to smear fields and yet avoid the introduction of auxiliary metrics. Back to the scalar, we can choose to smear p_ϕ instead of ϕ. The momentum of a scalar field is a scalar density; it transforms under coordinate changes with an extra factor of the Jacobian for the coordinate transformation. This is a direct consequence of the definition $p_\phi = \partial\mathcal{L}/\partial\dot{\phi}$ as a derivative of the Lagrangian density. Explicitly, the canonical variable $p_\phi = \sqrt{\det q}\dot{\phi}$ already carries the correct measure factor which need not be introduced by an auxiliary metric.[7] The smeared version $p_\phi[\lambda] := \int \mathrm{d}^3y\lambda(y)p_\phi(y)$ is well-defined for any function λ, and it suffices to remove delta functions from the Poisson algebra: $\{\phi(x), p_\phi[\lambda]\} = \lambda(x)$.

3.1.2 Holonomies and fluxes

For tensorial fields as we have them in gravity, background-independent smearings are often more difficult to find. Loop quantum gravity has provided suitable procedures for general relativity, but for this it must first transform from metric variables to connections with their conjugates, densitized triads. Connections and densitized vector fields turn out to have just the right transformation properties under coordinate changes that they can, with one loopy trick, be smeared background independently. A well-defined quantization results with several immediate implications for the basic operators encoding spatial geometry, as well as far-reaching and sometimes surprising consequences in the resulting dynamics.

Instead of using the spatial metric q_{ab}, spatial geometry is expressed by a densitized triad $E^a_i = \sqrt{\det q}e^a_i$ such that $E^a_iE^b_i = \det q\, q^{ab}$. The densitized triad is canonically conjugate to $K^i_a := K_{ab}e^{bi}$ in terms of extrinsic curvature K_{ab}. To obtain a connection with its useful transformation properties, we finally follow Ashtekar (1987) and Barbero G. (1995) to introduce the Ashtekar–Barbero connection $A^i_a = \Gamma^i_a + \gamma K^i_a$ with the spin connection Γ^i_a compatible with the densitized triad and a positive real number γ, the Barbero–Immirzi parameter (whose role for quantum geometry was realized by Immirzi (1997)).

The elementary objects loop quantum gravity takes for its representation are holonomies and fluxes,

$$h_e(A) = \mathcal{P}\exp\left(\int_e \mathrm{d}s\dot{e}^aA^i_a\tau_i\right) \quad \text{and} \quad F_S(E) = \int_S \mathrm{d}^2yn_aE^a_i\tau_i\,,$$

first used in this context by Rovelli and Smolin (1990). A key advantage is that their algebra under Poisson brackets is well-defined, free of delta-functions (unlike the algebra of fields), and yet independent of any background metric. Only the dynamical fiels A^i_a and E^a_i are used, together with kinematical objects such as curves $e \subset \Sigma$ and surfaces $S \subset \Sigma$ as well as the tangent vectors $\dot{e}^a$ and co-normals n_a to them, but no independent metric structure.[8] All spatial geometrical properties are reconstructed from E^a_i via fluxes, and

[7] Although $\sqrt{\det q}$ appears in the relationship between p_ϕ and $\dot{\phi}$, from the viewpoint of Poisson geometry p_ϕ (but not $\dot{\phi}$) is independent of the metric: $\{p_\phi, p^{ab}\} = 0$ for the momenta p^{ab} of q_{ab}.

[8] The co-normal, unlike the normal n^a, is metric independent: for a surface S: $f = \text{const}$, $n_a = (\mathrm{d}f)_a$.

space-time geometry follows with A_a^i via holonomies once equations of motion (or rather the constraints of relativity) are imposed. In this way, loop quantum gravity provides a framework for background-independent quantum theories of gravity.

Once a well-defined algebra of basic objects has been chosen, one can determine its representations to arrive at possible quantum theories. In the connection representation, a complete set of states $\psi(A_a^i)$ is generated by holonomies as multiplication operators acting on $\psi(A_a^i) = 1$. In the case of loop quantum gravity, this has an immediate and general consequence. Holonomies take values in SU(2), and fluxes, depending on the momenta E_i^a, become derivative operators on SU(2) — just like angular momentum in quantum mechanics. For a dense set of states, only a finite number of holonomies (along curves intersecting the flux surface S) contribute to a given flux. Finite sums of angular momentum operators with discrete spectra have a discrete spectrum, too: spatial geometry is discrete; flux operators quantizing the densitized triad and thus encoding spatial geometry acquire discrete spectra. So do spatial geometrical quantities such as areas and volumes as constructed by Rovelli and Smolin (1995); Ashtekar and Lewandowski (1997, 1998). No extra assumptions are required; one merely has to fix the basic algebra and follow mathematical procedures to analyze its representations. A different algebra might lead to other properties, possibly not with discrete spatial geometry. But no alternative procedure providing a well-defined and smeared, yet background independent quantization has been found. The holonomy-flux algebra suggested by its natural smearing, on the other hand, has a unique irreducible, cyclic, diffeomorphism covariant representation as proven by Lewandowski et al. (2006) andFleischhack (2009). Most of these properties are described by Sahlmann (2010).

3.1.3 Dynamics

Kinematical properties are elegant, simple, and largely unique. The theory starts to get considerably more messy when its dynamics is considered. Here, two major tasks must be performed: Dynamical operators, mainly the Hamiltonian constraint, must be defined from the basic ones, holonomies and fluxes. This is the constructive part of the task. For suitable constructions of the Hamiltonian constraint, the dynamics of the theory must then be evaluated, a process still full of several open issues.

At the constructive stage, many choices can be made, and strong consistency conditions must be respected. We are witnessing an epic battle between the liberating anarchy of choice and the uniformizing tyranny of constraints. Just writing a Hamiltonian constraint operator is tedious but possible in many ways. There are ubiquitous factor ordering ambiguities, as well as other choices specific to loop quantum gravity. On the other hand, the Hamiltonian constraint provides not only an equation of motion, it also generates a crucial part of the gauge transformations responsible for general covariance. A consistent quantization must keep gauge degrees of freedom as gauge, and not overly restrict the number of physical, non-gauge degrees of freedom. All this can usefully be formalized, as we will see later, providing strong algebraic conditions. They are so strong that to date, despite the abundant freedom of choices in constructing Hamiltonian constraint operators,

no full consistent version has been found.

Once a consistent version of the dynamics exists, it must be evaluated. We must find solutions, and determine the physical observables they provide. Their values, finally, can be used for predictions such as small deviations from the expected classical behavior. Since no full consistent version has been found yet, and since all potential condidates are highly complicated, construction issues of the dynamics have so far dominated strongly over evaluation issues. In several model systems, on the other hand, the dynamics can often be simplified so much that it can be analyzed rather explicitly. Many useful techniques are now available, mainly in the context of effective equations. We will come back to these applications after discussing more details of the construction side of the problem.

3.1.4 Hamiltonian constraint

Specifically, the Hamiltonian constraint of general relativity in Ashtekar variables is

$$C[N] = \int_\Sigma \mathrm{d}^3x N(x) \left(\epsilon_{ijk} F^i_{ab} \frac{E^a_j E^b_k}{\sqrt{|\det E|}} - 2(1+\gamma^{-2}) K^i_a K^j_b \frac{E^{[a}_i E^{b]}_j}{\sqrt{|\det E|}} \right)$$

with the curvature F^i_{ab} of the Ashtekar connection and extrinsic curvature K^i_a. It has to vanish for all lapse functions $N(x)$, thus providing infinitely many constraints. If $C[N]$ is to be turned into an operator, using the basic expressions for holonomies and fluxes, several obstacles must be overcome.

First, there is the potentially singular inverse determinant of the densitized triad. No direct quantization exists since the densitized triad has been quantized to flux operators with discrete spectra, containing zero. Such operators lack densely defined inverses. Nevertheless, quantizations with the appropriate inverse as the semiclassical limit can be obtained making use of the classical identity

$$\left\{ A^i_a, \int \sqrt{|\det E|} \mathrm{d}^3x \right\} \propto \epsilon^{ijk} \epsilon_{abc} \frac{E^b_j E^c_k}{\sqrt{|\det E|}} \tag{14}$$

(or variants) as introduced by Thiemann (1998a). There is no inverse on the left-hand side. Instead, the expression involving the densitized triad is the spatial volume which (with some regularization) can directly be quantized. The connection components can be expressed via holonomies, and the Poisson bracket will, at the quantum level, be quantized to a commutator divided by $i\hbar$. A well-defined operator results with the right-hand side of (14) as the desired semiclassical limit by construction.

The remaining factors in the Hamiltonian constraint involve the Ashtekar curvature as well as extrinsic curvature. For the curvature components F^i_{ab}, we can use

$$s^a_1 s^b_2 F^i_{ab}(x) \tau_i = \Delta^{-1}(h_\lambda - 1) + O(\Delta) \tag{15}$$

where λ is a small loop starting at a point x, spanning a coordinate area Δ, and with tangent vectors s^a_1 and s^a_2 at x. On the right-hand side, the holonomy h_λ can readily

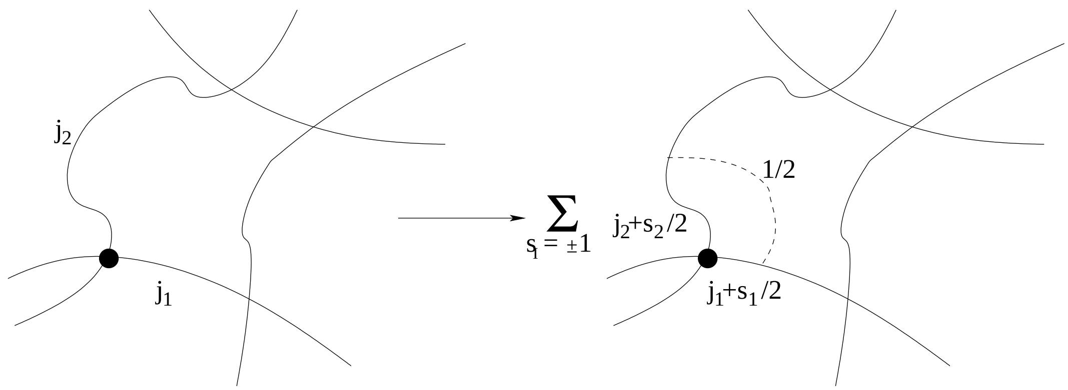

Figure 1: Schematically, the local action of the Hamiltonian constraint operator on a state. States are generated by holonomies as multiplication operators, visualized by the graph formed by all curves e used. Moreover, labels j_e determine matrix representations of SU(2)-valued $h_e(A)$. Due to (15), new curves and vertices are typically created when the Hamiltonian constraint acts.

be quantized, and to leading order provides curvature components as required for the constraint.

Extrinsic curvature, finally, is a more complicated object in terms of the basic ones but can be obtained from what has been provided so far:

$$K_a^i \propto \left\{ A_a^i, \left\{ \int \mathrm{d}^3x F_{ab}^i \frac{\epsilon^{ijk} E_j^a E_k^b}{\sqrt{|\det E|}}, \int \sqrt{|\det E|} \mathrm{d}^3x \right\} \right\}$$

expresses extrinsic curvature in terms of a nested Poisson bracket involving the spatial volume and the first term of the constraint, already provided by the preceding steps.

In this way, holonomy and flux operators make up the Hamiltonian constraint operator $\hat{C}$ as a densely defined operator (including, in the non-vacuum case, regular matter Hamiltonians again exploiting (14) following Thiemann (1998b)). It determines the physical solution space by its kernel: physical states $\psi(A)$, assumed again in the connection representation, must satisfy $\hat{C}\psi(A) = 0$, or $(\hat{C} + 8\pi G \hat{H}_{\mathrm{matter}})\psi(A, \phi, \ldots) = 0$ if matter is present. The action of the constraint is rather complicated, as visualized schematically in Fig. 1

Physical states of interest normally belong to zero in the continuous part of the spectrum of $\hat{C}$, which requires the introduction of a new physical Hilbert space spanned by the solutions to the quantum constraint equation and equipped with a suitable physical inner product. Constructing the physical Hilbert space leads over to evaluating the dynamics, for physical states would provide predictions from expectation values of observables. But this stage has been brought to completion only in a few simple (and very special) models; in general the outlook toward a full implementation is rather pessimistic.

Here, effective constraint techniques become useful because they allow one to address

physical properties, corresponding to observables in the physical Hilbert space without having to deal explicitly with states. Conditions from the physical inner product are rather implemented by reality conditions for expectation values and quantum variables such as fluctuations, which can be done more simply and more generally than for entire states. Effective constraints thus provide a good handle on generic properties of physical observables, at least in semiclassical regimes where not all the moments need be considered. Especially in cosmological situations they provide an ideal framework. In inhomogeneous contexts, they allow a detailed discussion of the anomaly issue, and show whether the different effects expected from the basic operators of quantum gravity can lead to a consistent form of the dynamics.

Consistency conditions in the presence of ambiguities are useful, for they constrain the choices. But the question remains whether loop quantum gravity can be fully consistent at all — implementing covariance at the quantum level might, after all, not leave any consistent physical states. If the consistency conditions are weak, on the other hand, ambiguities would remain even at the physical level. In between, at a razor-thin balance between ambiguity and constraints, lies the case of a unique consistent theory, a possibility which exists in loop quantum gravity but for which at present no evidence has been found.

In model systems it has at least been shown that consistency in the form of covariance can be achieved even in the presence of quantum corrections resulting from the discreteness. This statement may come as a surprise, for discrete spatial or space-time structures are naively expected to break local Lorentz or even rotational symmetries. Nevertheless, at the level of effective equations one can see that covariance can be respected — but it cannot leave the classical algebra of local symmetries invariant. While quantum-gravity corrections provide a consistent deformation of general relativity — preserving the number of independent gauge generators — the algebra is truly deformed. Gauge transformations are no longer local Lorentz or coordinate transformations, but of a different type relevant for the quantum space-time structures derived from loop quantum gravity.

Much of these conclusions makes use of results obtained through several years for models of loop quantum cosmology. We will now start a general exposition beginning with the simplest isotropic models, introducing the key features (and sources of ambiguities), and then leading over to the discussion of covariance in the next section.

3.2 Isotropic loop quantum cosmology

Loop quantum cosmology provides quantizations of symmetry-reduced models of general relativity, starting with isotropic ones in the simplest cases. It is thus a minisuperspace quantization, but not only that. There are explicit relationships between the states and the algebra of basic operators in a model of loop quantum cosmology, which shows how they descend from the analogous expressions in the full theory. For different kinds of reductions such as implementing isotropy, homogeneity or spherical symmetry, this has been constructed by Bojowald et al. (2006); Bojowald (2004, 2006). In particular, properties of the basic operators, for instance the discreteness of the flux spectra, are preserved and realized also in cosmological models. Crucial implications for the discreteness of spatial geometry

can then be analyzed in their dynamical context. Qualitatively, all applications of loop quantum cosmology rely on this preservation of discreteness properties by the symmetry reduction introduced by Bojowald and Kastrup (2000).

While existing methods do not allow one to derive the dynamics of models directly from full constraint operators, it can be constructed in an analogous way using all the construction steps sketched for the full constraint. Also here, crucial properties are preserved. And even though the incompletely known relationship to the full theory introduces additional ambiguities in the dynamics of models, reliable conclusions can still be drawn provided the considerations are generic enough. Here, the need for sufficiently general parameterizations of ambiguities arises.

3.2.1 Basic variables

In spatially flat isotropic models, the basic canonical fields reduce to $A_a^i = \tilde{c}\delta_a^i$ and $E_i^a = \tilde{p}\delta_i^a$ with two dynamical variables $\tilde{c}$ and $\tilde{p}$. Relating the densitized triad to the spatial metric shows that $|\tilde{p}| = a^2$ is given by the scale factor. (Due to the freedom of choosing an orientation of the triad, $\tilde{p}$ can take both signs.) Classically, $\tilde{c} = \gamma\dot{a}$ for spatially flat models.

To define the Poisson structure, we pull back the full symplectic form $(8\pi\gamma G)^{-1}\int \mathrm{d}^3x\delta A_a^i \wedge \delta E_i^a$ by the embedding $(\tilde{c},\tilde{p}) \mapsto (A_a^i, E_i^a) = (\tilde{c}\delta_a^i, \tilde{p}\delta_i^a)$. Due to homogeneity, the resulting integral diverges if we integrate over all of space and space is infinite, but homogeneity also implies that we may just integrate over any finite chunk of coordinate volume V_0, say, and still get the complete reduced symplectic form, $3V_0(8\pi\gamma G)^{-1}\mathrm{d}\tilde{c}\wedge\mathrm{d}\tilde{p}$. It depends on the arbitrary V_0, which can be hidden in the canonical variables by redefining them as

$$c = V_0^{1/3}\tilde{c} \quad \text{and} \quad p = V_0^{2/3}\tilde{p}\,.$$

In the end, physical quantities must be ensured to depend only on combinations of c, p and possibly other ingredients such that they are insensitive to changes of V_0. (Changing V_0 is independent of changing coordinates, thereby rescaling the scale factor. While $\tilde{c}$ and $\tilde{p}$ depend on the scaling but not on V_0, c and p depend on V_0 but not on the scaling.)

Isotropic connections and densitized triads result in specific versions of holonomies and fluxes. They take especially simple forms when evaluated along curves or surfaces making use of structures provided by the homogeneity group, i.e. curves along generators of translations and surfaces transversal to them. In a homogeneous context, fluxes are simply the triad components multiplied with the coordinate area of the surface, and holonomies can be computed explicitly: $h_{e_j}(A_a^i) = \exp(\mu c\tau_j)$ with $\mu = \ell_0/V_0^{1/3}$ for a curve e_j of coordinate length ℓ_0 along the direction $\dot{e}_j^a = (\partial/\partial x^j)^a$ in Cartesian coordinates. If we take surfaces for fluxes of edge length the same size as e_j, fluxes are $F_{S_j}(E_i^a) = \mu^2 p\tau_j$ for a surface S_j transversal to $(\partial/\partial x^j)^a$.

These curves and surfaces are particularly useful because they can be arranged in a regular lattice of spacing ℓ_0. While such a setting is not the most general one (and other curves would lead to different expressions for holonomies as computed for instance by Brunnemann and Fleischhack (2007)), it allows one to capture all the degrees of freedom

in a homogeneous model. More importantly, it allows for rough contact with the full theory, for the regular lattices envisioned here can be thought of as lattices for states in the full theory. In particular for comparisons of actions of the dynamical operators, such a relationship is convenient. It also allows us to find a useful interpretation of the parameter μ characterizing isotropic holonomies and fluxes: For a regular lattice of spacing ℓ_0, contained in a region of size V_0, $\mu^3 = \ell_0^3/V_0 =: \mathcal{N}^{-1}$ is the inverse number of lattice sites. This parameter can be seen as providing information about an underlying discrete inhomogeneous state which appears even in a homogeneous reduction. In fact, properties of $\mathcal{N}$ such as its size or possible dynamical features play a crucial role in Hamiltonian quantum evolution.

3.2.2 Representation

From isotropic holonomies and fluxes, we can construct their loop representation by analogy with the full theory. States in the connection representation are functionals of holonomies and are thus superpositions of the basic states $|\mu\rangle$, or $\exp(i\mu c)$ as functions of the isotropic connection component. Here, μ is an arbitary real parameter. An inner product of states can be derived from integration theory on spaces of connections, as developed for instance by Ashtekar et al. (1995). For isotropic models, this makes all states $|\mu\rangle$ orthonormal.

Upon completion to a Hilbert space, a general state takes the form of a countable superposition of $\exp(i\mu c)$ with $\mu \in \mathbb{R}$:

$$\psi(c) = \sum_{I\in\mathcal{I}\subset\mathbb{R},\ \text{countable}} f_I \exp(i\mu_I c) \tag{16}$$

such that $\sum_{I\in\mathcal{I}} |f_I|^2$ exists. These states form a Hilbert space equivalent to the ℓ^2-space formed by the normalizable sequences $(f_I)_{I\in\mathcal{I}}$ for all countable $\mathcal{I} \subset \mathbb{R}$. In this form, isotropic states directly result from the reduction of a full state, where all the holonomies reduce to exponentials of c with different exponents. Alternatively, the Hilbert space may be characterized as the space of square integrable functions $\psi(c)$ on the Bohr compactification of the real line, a compact space containing the real line densely and equipped with the measure

$$\int \mathrm{d}\nu(c)\psi(c) = \lim_{C\to\infty} \frac{1}{2C} \int_{-C}^{C} \psi(c)\mathrm{d}c$$

using the normal Lebesgue measure on the right-hand side. This Hilbert space is non-separable.

Basic operators act by multiplication or differentiation:

$$\widehat{\exp(i\delta c)}|\mu\rangle = |\mu+\delta\rangle \tag{17}$$

$$\hat{p}|\mu\rangle = \frac{8\pi}{3}\gamma\ell_{\mathrm{P}}^2\mu|\mu\rangle \tag{18}$$

as it indeed follows from the unique holonomy-flux algebra in the full theory as an induced representation. A Wheeler–DeWitt representation, by contrast, would not be related to such a representation of the full theory.

Properties of the loop representation are markedly different from the Wheeler–DeWitt one, but they closely mimick properties of the full holonomy-flux representation:

- There is a discrete spectrum of $\hat{p}$. While there is a continuous range for μ, all eigenstates $|\mu\rangle$ are normalizable. For a non-separable Hilbert space as we are dealing with here, normalizability of eigenstates does not imply that the eigenvalues form a countable subset of the real line. In such a situation, the normalizability condition is more general, for it is insensitive to what topology one uses on the set of eigenvalues. Even if any real number can appear as an eigenvalue, this set would be discrete if one uses a discrete topology of the real line (for instance, one where every subset is an open neighborhood). Normalizability of eigenstates will also be one of the crucial properties for consequences of flux spectra in loop quantum cosmology.

- There is no operator for c, and only holonomies are represented. Trying to derive an operator for c from holonomies, for instance by taking a derivative of the action of $\widehat{\exp(i\delta c)}$ by δ at $\delta = 0$, fails because holonomies are not represented continuously in δ: $\langle\mu|\widehat{\exp(i\delta c)}|\mu\rangle = \langle\mu|\mu+\delta\rangle = \delta_{\delta,0}$ is not continuous.

Ashtekar et al. (2003) and Fewster and Sahlmann (2008) discuss these and other properties of the Bohr compactification as used in loop quantum cosmology, and Husain and Winkler (2004) do so in a quantum cosmology based on ADM variables.

Since these are the same basic properties as they are realized for the full holonomy-flux algebra, they have the same qualitative implications for the dynamics. Once used for the construction of operators such as the Hamiltonian constraint, they lead to specific quantum-geometry corrections in loop quantum cosmology as they do in the full theory. Specifically,

1. Only almost-periodic functions of c are represented as operators on states (16), and must be expressible as $\psi(c) = \sum_{I\in\mathcal{I}\subset\mathbb{R},\mathrm{countable}} f_I \exp(i\delta_I c)$. The Hilbert space does not allow an action of c on its dense subset of triad eigenstates; any appearance of c not of almost-periodic form, such as the polynomial in the isotropic Hamiltonian constraint, must be expressed in terms of almost-periodic functions by adding suitable higher-order corrections in c. They become significant for large values of the curvature, via $\ell_0\tilde{c} = c/\mathcal{N}^{1/3}$ for a regular distribution of edges where $\delta_I \sim \mathcal{N}^{-1/3}$.

2. The isotropic flux operator $\hat{p}$ has a discrete spectrum containing zero and so lacks a direct, densely defined inverse. Well-defined versions can be obtained via identities such as
$$\frac{i}{\delta}e^{i\delta c}\{e^{-i\delta c}, |p|^{r/2}\} = \{c, |p|^{r/2}\} = \frac{4\pi\gamma G r}{3}|p|^{r/2-1}\,\mathrm{sgn}p \tag{19}$$
which mimick the crucial one (14) used in the full theory. For $0 < r < 2$ we are expressing an inverse of p on the right-hand side, but do not need an inverse on the left-hand side; well-defined and even bounded operators for inverse powers of p result, as derived by Bojowald (2001b). Within this range, r is unrestricted by

general considerations; it thus appears as an ambiguity parameter (see Bojowald (2002b) for further discussions of ambiguities). Also here, corrections to classical expressions arise, in this case for small flux values $\ell_0^2\tilde{p} = p/\mathcal{N}^{2/3}$ near the Planck scale. From a quantization of the left-hand side of (19), eigenvalues can readily be derived. They have the form $\frac{4}{3}\pi\gamma G r|p|^{r/2-1}\alpha_r(p)$ where $\alpha_r(p) \sim 1$ asymptotically for large $p/\mathcal{N}^{2/3} \gg \ell_{\mathrm{P}}^2$ while $\alpha_r(p) \to 0$ rapidly for $p \to 0$, cutting off the divergence of $|p|^{r/2-1}$. The correct semiclassical limit is guaranteed by α_r approaching one, which it does from above irrespective of quantization ambiguities.

3.2.3 Difference equation

The main quantity for which these considerations play a role is the Hamiltonian constraint, in isotropic variables providing the Friedmann equation

$$\frac{c^2\sqrt{|p|}}{\gamma^2} = \frac{8\pi G}{3} H_{\mathrm{matter}}$$

with the matter Hamiltonian $H_{\mathrm{matter}} = V_0 a^3\rho$. Classically, its dependence on c is via c^2, which is not almost periodic. There are many ways to express c^2 in terms of almost-periodic functions such that they approximate c^2 for small curvature, $c \ll 1$. In general terms, properties of the quantum representation space require the replacement of c^2 to be a bounded function of c, given by a normalizable superposition $\exp(\delta_I c)$ with δ_I in a countable subset of the real line. Each exponential acts by a shift of triad eigenvalues, rather than a derivative operator, and so, following Bojowald (2001c, 2002a), upon loop quantization the Hamiltonian constraint equation becomes a difference equation for components of a wave function in the triad representation.

A choice usually made is $c^2 \sim \delta^{-2}\sin^2(\delta c)$, but others are possible, constituting further ambiguities. A Hamiltonian constraint operator

$$\hat{C}_{\mathrm{iso}} = \frac{3}{8\pi G\gamma^2\delta^2}\widehat{\sin\delta c}^{\,2}\widehat{\sqrt{|p|}} - \hat{H}_{\mathrm{matter}}$$

results. Expanding a state $|\psi\rangle = \sum_\mu \psi_\mu|\mu\rangle$ in the triad eigenbasis $\{|\mu\rangle\}_{\mu\in\mathbb{R}}$, $\hat{C}_{\mathrm{iso}}|\psi\rangle = 0$ is equivalent to the difference equation

$$C(\mu+2\delta)\psi_{\mu+2\delta} - 2C(\mu)\psi_\mu + C(\mu-2\delta)\psi_{\mu-2\delta} = \frac{8\pi G}{3}\gamma^2\delta^2\hat{H}_{\mathrm{matter}}(\mu)\psi_\mu \tag{20}$$

as derived by Bojowald (2002a), where $C(\mu)$ are eigenvalues of $\widehat{\sqrt{|p|}}$, e.g. proportional to $\sqrt{|\mu|}$. Additional matter fields have been suppressed in the notation, which would provide further independent variables in ψ_μ acted on by the matter Hamiltonian $\hat{H}_{\mathrm{matter}}(\mu)$. (Unless there are curvature couplings, matter Hamiltonians depend on the densitized triad but not on the connection; the right-hand side of (20) is then not a difference expression. Fermions, which require a coupling to the gravitational connection, have been discussed in this context

by Bojowald and Das (2008), and non-minimally coupled scalars by Bojowald and Kagan (2006).)

Due to the inequivalence of representations, this difference equation replaces the differential Wheeler–DeWitt equation. At this dynamical level, after choosing the representation of c^2 by almost-periodic functions, the dynamics can be restricted to a separable subsector of the initial non-separable Hilbert space. From this perspective, one could have started directly from a separable Hilbert space by choosing a specific countable set $\{\mu_I\} \subset \mathbb{R}$, such as $\mu_I = I\mu_0$ with integer I as originally defined by Bojowald (2002a). Many dynamical properties would follow in the same way since non-separable features appear only at the kinematical level.

However, there is a good reason for dealing with a non-separable kinematical Hilbert space, allowing for all real values of μ: To capture the most general and viable dynamics, one expects lattice refinement to happen, with a discreteness scale changing dynamically. Indeed, full Hamiltonian constraint operators create new edges and vertices, changing the graph on which a state is defined, as illustrated in Fig. 1. In an isotropic context, a trace of this feature must still be left since the dynamical parameter δ came from the edge length ℓ_0 used in elementary holonomies. While V_0 in $\delta = \ell_0/V_0^{1/3}$ remains constant as a mere auxiliary parameter introduced in the setup, ℓ_0 must be adapted to the geometry. Geometrical distances, after all, increase as the universe expands and the region of coordinate size V_0 has a growing geometrical size $V_0 a^3$. If a lattice state for a larger region has more sites, as suggested by the creation of vertices, ℓ_0 must shrink as a grows; otherwise the discrete size $a\ell_0$ would be blown up macroscopic. Here, the evolutionary aspects capture expected properties of full physical states, solving a vertex-creating Hamiltonian constraint, arranged by volume eigenvalues as internal time. (From a more general perspective, dynamical lattices and their role in quantum gravity have been discussed by Weiss (1985); Unruh (1997); Jacobson (2000).) Then, also $\delta(\mu)$ must not be constant but rather a decreasing function. (Just as in (15), for the regularization of the isotropic F^i_{ab}, or the c^2 in the Friedmann equation, by $\delta^{-2}\sin^2(\delta c)$, we use the coordinate area ℓ_0^2 of loops, not the geometrical area $\ell_0^2 a^2$ which might well be constant as a changes.)

In summary, lattice refinement means that δ depends on the size of the universe, and $\delta(\mu)$ is not a constant. The resulting difference equations, as most generally formulated for anisotropic models by Bojowald et al. (2007c), are not equidistant and do not allow simple restrictions to separable sectors. A large class of different dynamics can thus be formulated in each single model.[9] Consistency conditions do exist restricting the freedom even in homogeneous models, but systematic investigations have only just begun, as e.g. by Bojowald (2009). While specific details of the difference equations of loop quantum cosmology are not yet fully determined, owing to quantization ambiguities, there are several key generic features implying further properties. They are brought out most clearly in sovable models.

[9]Several improvised models have been formulated and sometimes analyzed in detail, based on further ad-hoc assumptions to constrain ambiguities. Since such an approach cannot capture the generic behavior, specific details of the results may not be reliable.

3.2.4 Harmonic cosmology

Sometimes, solutions to the difference equations of loop quantum cosmology, which are linear but with non-constant coefficients, can be found. They may be studied numerically, and lead to interesting issues especially in cases where they are not equidistant as investigated so far by Sabharwal and Khanna (2008); Nelson and Sakellariadou (2008). But in all cases, computing observables from the resulting wave functions and arriving at sufficiently generic conclusions is challenging.

An effective treatment at this stage becomes much more powerful. Initially, one may expect technical difficulties due to the non-linear and non-polynomial nature of the Hamiltonian constraint obtained after the loop replacement of c^2. Fortunately, isotropic loop quantum cosmology offers an exactly solvable system where, in a specific factor ordering of the constraint operator, the dynamics is essentially free. From the point of view of quantum dynamics, this model is closely related to the harmonic oscillator of quantum mechanics, which forms the basis of most of the perturbation-theory framework for interacting quantum field theories. Similarly, the harmonic model of loop quantum cosmology plays a crucial role for perturbations in quantum gravity.

Specifically, according to Bojowald (2007b,a), we obtain the solvable model for spatially flat isotropic loop quantum cosmology with a free, massless scalar, whose loop-modified Hamiltonian constraint is

$$\frac{4\pi G}{3}\frac{p_\phi^2}{p^{3/2}} = \sqrt{p}\gamma^{-2}\delta(p)^{-2}\sin^2(\delta(p)c) \sim \sqrt{p}c^2 + O(c^4)\,.$$

Here, $\delta(p)$ (or $\mathcal{N}(p) = \delta(p)^{-3}$) is allowed to depend on the triad, incorporating different refinement schemes of the discrete structure. For power laws $\delta(p) \propto p^x$, we introduce $V = p^{1-x}/(1-x)$ and $J = p^{1-x}\exp(ip^x c)$ as non-canonical basic variables forming an $\mathrm{sl}(2,\mathbb{R})$ algebra

$$[\hat{V},\hat{J}] = \hbar\hat{J} \quad , \quad [\hat{V},\hat{J}^\dagger] = -\hbar\hat{J}^\dagger \quad , \quad [\hat{J},\hat{J}^\dagger] = -2\hbar\hat{V}$$

upon quantization. Thanks to the free scalar, the dynamics is controlled by the deparameterized Hamiltonian in ϕ, $\hat{p}_\phi \propto |\frac{1}{2i}(\hat{J}-\hat{J}^\dagger)| =: \hat{H}$.

This Hamiltonian is linear for all x, which for our linear algebra of non-canonical basic variables implies solvability. (Again, the absolute value is irrelevant for effective equations.) Following our general analysis of effective descriptions, we can directly jump to equations of motion for expectation values of the basic variables. They are coupled to each other, but not to quantum variables such as fluctuations as it normally occurs in interacting quantum systems. Instead of a non-linear set of infinitely many differential equations, we have a finitely coupled set of linear equations. For the basic variables, we have

$$\begin{aligned}
\frac{\mathrm{d}\langle\hat{V}\rangle}{\mathrm{d}\phi} &= \frac{\langle[\hat{V},\hat{H}]\rangle}{i\hbar} = -\tfrac{1}{2}(\langle\hat{J}\rangle + \langle\hat{J}^\dagger\rangle)\\
\frac{\mathrm{d}\langle\hat{J}\rangle}{\mathrm{d}\phi} &= \frac{\langle[\hat{J},\hat{H}]\rangle}{i\hbar} = -\tfrac{1}{2}\langle\hat{V}\rangle = \frac{\mathrm{d}\langle\hat{J}^\dagger\rangle}{\mathrm{d}\phi}\,.
\end{aligned}$$

As a difference to the previous discussion of effective systems, we now have to take into account the complex nature of our variable J. Reality conditions are required for physical results, which would be implemented by the physical inner product in a Hilbert-space representation. Here, we are not using states but directly work with expectation values and moments. The adjointness relation $\hat{J}\hat{J}^\dagger = \hat{V}^2$ for our basic operators implies, upon taking an expectation value, a reality condition relating $|\langle\hat{J}\rangle|^2 - \langle\hat{V}\rangle^2$ to moments of $\hat{V}$ and $\hat{J}$. It turns out that the specific combination of moments involved is constant in ϕ-evolution, and that it is of the order $\hbar$ for a semiclassical state. (For more details, see Bojowald (2007a, 2008b).) Just requiring that the state is semiclassical only once, for instance at large volume, ensures that the reality condition reads $|\langle\hat{J}\rangle|^2 - \langle\hat{V}\rangle^2 = O(\hbar)$ at all times. With this condition, the general solution is

$$\langle\hat{V}\rangle(\phi) = \langle\hat{H}\rangle\cosh(\phi-\beta) \quad , \quad \langle\hat{J}\rangle(\phi) = -\langle\hat{H}\rangle(\sinh(\phi-\beta)-i)$$

with the conserved $\langle\hat{H}\rangle$ and another integration constant β. For a large class of states we thus have an exact realization of a bounce, with the volume bounded away from zero. (Effective equations thus easily confirm the numerical results of Ashtekar et al. (2006), at least as far as physical expectation values are concerned. For the dynamical behavior of moments, the generic treatment of Bojowald (2007c, 2008b) based on effective equations shows crucial differences to the numerics done for only a specific class of states.)

However, the system is harmonic, and so its behavior is not easily generalizable to realistic models. While it is interesting that bounce models can be derived in this way, taking this at face value without realizing the solvable nature of the model would provide a view with severe limitations. After all, the harmonic oscillator does not provide general insights into quantum dynamics, and free quantum field theories allow no glimpse at the rich features of interacting ones. Only a systematic analysis of perturbations around the free model, starting with interaction terms and then introducing inhomogeneities, can provide a clear picture.

3.2.5 Quantum Friedmann equation

As already briefly discussed in the context of Wheeler–DeWitt quantum cosmology, effective equations of quantum cosmology are necessarily of higher dimension than the classical equations. Quantum degrees of freedom such as fluctuations couple to expectation values in non-solvable models, and no adiabatic or other regime has been determined where quantum variables could completely be expressed in terms of effective contributions depending only on the expectation values. Quantum variables are true degrees of freedom, of significance for the dynamics. Without a clear vacuum state to expand around, moreover, suitable states are less restricted than they are for low-energy effective actions. Via initial values for moment equations, this leaves the state dependence as a crucial contribution of equations, to be taken into account for sufficiently general conclusions. Since not much is known about details of the quantum state of the universe, results should be sufficiently insensitive to its properties.

Perturbation theory by the moments, including them order by order, is required for any model which is not harmonic. Examples are models with non-vanishing spatial curvature or a cosmological constant as analyzed by Bojowald and Tavakol (2008) (or numerically by Bentivegna and Pawlowski (2008)). Another class of important models is that where the scalar is no longer free or at least acquires a mass term. We are then dealing with a time-dependent Hamiltonian in the ϕ-evolution, and ϕ may not even provide a good internal time if the potential leads to turning points of $\phi(\tau)$. At this stage, a more general analysis is required which is now available in the form of effective constraints as per Bojowald et al. (2009a) and Bojowald and Tsobanjan (2009): we do not have to deparameterize the system before quantizing it or computing effective equations. We can directly compute effective constraints and analyze them to find the physical quantum phase space. As one of the results, we can apply the deparameterized framework even if there is a potential provided it changes sufficiently slowly. This is exactly the case of interest for inflationary or other early-universe cosmology, and so we can derive effective equations for these situations.

In the presence of a potential, the Friedmann equation receives the following quantum corrections:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho\left(1-\frac{\rho_Q}{\rho_{\rm crit}}\right) \pm \frac{1}{2}\sqrt{1-\frac{\rho_Q}{\rho_{\rm crit}}}\,\eta(\rho-P) + \frac{(\rho-P)^2}{\rho+P}\eta^2\right)$$

as derived by Bojowald (2008d,b). (The first term, corresponding only to the solvable model with a free scalar, was earlier obtained by Singh and Vandersloot (2005); Singh (2006). It sums up all higher-order corrections due to holonomies, as explicitly expanded by Banerjee and Date (2005). In the usual terminology of quantum field theory, it represents the tree-level approximation since its main form is independent of quantum back-reaction.) Here, P is pressure, η parameterizes quantum correlations and

$$\rho_Q := \rho + \epsilon_0\rho_{\rm crit} + (\rho-P)\sum_{k=0}^{\infty}\epsilon_{k+1}\left(\frac{\rho-P}{\rho+P}\right)^k$$

is a quantum-corrected energy density with fluctuation parameters ϵ_k. The critical density $\rho_{\rm crit} = 3/8\pi G\gamma^2\delta^2 V_0^{2/3}a^2$ results from the loop quantization, bringing in the scale $\delta(a)$ whose form is determined by lattice refinement.

This equation contains all quantum variables in η and ρ_Q, subject to their own dynamics. Understanding the behavior of generic universe models in loop quantum cosmology requires a high-dimensional dynamical system to be analyzed. Only a few simple cases allow strict conclusions about the presence of a bounce. If $\rho = P$ (a free, massless scalar) we recover the solvable scenario. For $\rho + P$ large (large p_ϕ, i.e. kinetic domination), all corrections involving quantum variables are supressed; at least for a certain amount of time, the behavior can be expected to be close to the solvable one.

In general, however, quantum variables may yank the universe away from the simple bouncing behavior of the solvable model. While the fundamental dynamics based on difference equations for states remains non-singular, following Bojowald (2001a), the general effective one is still unclear. Volume expectation values could asymptotically approach a

non-zero constant without bouncing back to large values, a picture which might resemble that of the emergent universe of Ellis and Maartens (2004); Ellis et al. (2004). Most likely, generic evolution would bring us to a highly quantum regime where effective equations or simple near-solvable models break down. Ultimately, the fate of a universe can be studied only based on the fundamental difference equations, but effective equations can show well how such severe states are approached.

3.2.6 The interplay of different quantum corrections

Loop quantum cosmology leads to different types of quantum corrections in its effective equations: holonomy corrections, inverse triad corrections, and quantum back-reaction. Depending on parameters, they may play different roles in any given regime, and some of them might dominate. What exactly happens can be determined only with a sufficiently general parameterization of correction terms. Here, all ambiguities and effects such as lattice refinement must be considered.

Especially holonomy and inverse triad corrections are often closely related to each other: they both result from the discrete geometry, although in different ways. Comparing them can thus provide more restrictions on free parameters of the full theory than any one of the corrections would allow individually. They are both related to the size of elementary building blocks of a discrete geometry. For a nearly isotropic distribution in a region of coordinate size V_0, the classical volume on the left-hand side of

$$V_0 a(\phi)^3 = \mathcal{N}(\phi) v(\phi)$$

is replaced by the right-hand side in the discrete picture. This elementary relationship first tells us that, dynamically, there are two free functions in the discrete picture for one free function $a(\phi)$ in the continuum picture: the size $v(\phi)$ and number $\mathcal{N}(\phi)$ of discrete sites. Both of them typically change in internal time: the lattice they describe is being refined as time goes on.

As derived earlier, one of these parameters, v, enters the basic expressions for holonomies

$$\exp(i\ell_0 c/V_0^{1/3}) = \exp(ic/\mathcal{N}^{1/3}) = \exp(i\gamma v^{1/3}\dot{a}/a)$$

using $c = V_0^{1/3}\tilde{c} = V_0^{1/3}\gamma\dot{a}$. Inverse-triad corrections in isotropic models provide a correction factor α depending on $p/\mathcal{N}^{2/3} = (V_0/\mathcal{N})^{2/3}a^2 = v^{2/3}$. For $v^{1/3} \sim \ell_{\mathrm{P}}$, inverse-triad corrections differ strongly from the classically expected value $\alpha = 1$ (while holonomies remain nearly equal to the classically linear $\dot{a}/a$ for Hubble distances much larger than $v^{1/3}$). Here, quantum corrections can often be constrained. Since two different corrections depend on one parameter, their interplay can provide important synergistic effects in ruling out possible values, or even entire corrections.[10]

[10]We can also note that all quantum geometry corrections depend only on $v = V_0a^3/\mathcal{N}$. As the size of discrete building blocks, it is insensitive to changing V_0 (both V_0 and $\mathcal{N}$ change proportionally to V_0) or coordinates (V_0a^3 and $\mathcal{N}$ are scaling invariant). Just like the classical equations, quantum corrections

Phenomenologically, v shows up in the critical density of the effective Friedmann equation: $\rho_{\rm crit} = 3/8\pi G\gamma^2 v^{2/3}$. At this value, we would find the bounce of a universe sourced by a free, massless scalar, and even in other models its value affects details of the dynamics. This density is often assumed Planckian, which means that $v \sim \ell_{\rm P}^3$. (From black-hole entropy calculations, as developed by Ashtekar et al. (1998), Kaul and Majumdar (1998), Domagala and Lewandowski (2004) and Meissner (2004), $\gamma \sim 0.2$.) But then, inverse triad corrections are large. If v is constant, which is also often assumed, inverse-triad corrections would be large at all times, even at large total volume. A Planckian constant value of v is thus clearly ruled out, not by holonomy corrections alone but by a consistent combination with inverse-triad effects. Further phenomenological analysis is in progress, by Nelson and Sakellariadou (2007a,b); Copeland et al. (2009); Mielczarek (2008); Barrau and Grain (2009); Grain et al. (2009b, 2010); Shimano and Harada (2009). Much stronger consistency conditions can be expected when inhomogeneities are included, to which we will turn next.

4 Consistent dynamics

A space-time covariant theory is a gauge theory with the gauge group given by space-time diffeomorphisms. As a consequence, the local conservation law $\nabla^\mu(G_{\mu\nu} - 8\pi G T_{\mu\nu})$ holds for the Einstein tensor $G_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$ of matter. Not all components of Einstein's equation are thus independent, and some can be derived from the others. For consistency, effective equations of quantum gravity must show the same form of dependencies, or allow the same number of local conservation laws, or else degrees of freedom would be too much constrained to have a chance of providing the correct classical limit.

Consistency is automatically satisfied if one has space-time covariant higher-order corrections in Lagrangian form. At the Hamiltonian level, however, covariance is more difficult to check due to the lack of a direct action of space-time diffeomorphisms. There is rather a splitting into spatial diffeomorphisms acting as usual, and an independent generator deforming spatial constant-time slices within space-time. As in relativistic systems, all generators are constraints, taking center stage in a canonical analysis: The conservation law implies that the components $G_{0\nu} - 8\pi G T_{0\nu}$ of Einstein's equation are only of first order in time derivatives since $\partial_0(G^0_\nu - 8\pi G T^0_\nu)$ must equal terms of at most second order in time. The same equation shows that these constraints are automatically preserved in time, given that $\partial_0(G^0_\nu - 8\pi G T^0_\nu)$ depends only on the Einstein equations at fixed time and their spatial derivatives. If Einstein's equation holds at one time, time derivatives of the components $G_{0\nu} - 8\pi G T_{0\nu}$ vanish; the constraints can consistently be imposed at all times. Also this condition must be preserved for effective equations. Otherwise, the system is

are invariant under these rescalings. For the scale factor, inverse-triad corrections become significant at a characteristic scale $a_* = (\mathcal{N}/V_0)^{1/3}\ell_{\rm P}$ related to the Planck length. While a_* is not scaling independent, it scales in the same way as a. Comparisons such as $a > a_*$ or $a < a_*$ to demarcate the classical and the strongly quantum regime are thus meaningful.

overdetermined with more equations than unknowns and not enough consistent solutions would result.

From this perspective, the covariance condition takes the following form at the Hamiltonian level. For a generally covariant system, evolution in coordinate time is generated by the constraints, and the constraints must themselves be preserved in time. Thus, the gauge transformation $\delta_{\epsilon^\mu} H[N^\nu] = \{H[N^\nu], H[\epsilon^\mu]\}$ of a combination $H[N^\nu] := \int \mathrm{d}^3x N^\nu (G_{0\nu} - 8\pi G T_{0\nu})$ must vanish for all space-time vector fields N^ν, ϵ^μ when the constraints are satisfied. These Poisson-bracket relations provide algebraic conditions for the constraints to be consistent, forming a so-called first-class system. Specifically for general relativity, we have

$$\{H[N^\mu], H[M^\nu]\} = H[K^\mu] \tag{21}$$

with $K^0 = \mathcal{L}_{M^a} N^0 - \mathcal{L}_{N^a} M^0$ and $K^a = \mathcal{L}_{M^a} N^a - \mathcal{L}_{N^a} M^a + q^{ab}(N^0 \partial_b M^0 - M^0 \partial_b N^0)$. It is a concise expression for space-time covariance, and is insensitive to the specific dynamics: the algebra is the same for all higher-curvature actions (or with all kinds of matter), even though the specific constraint functionals do change. (A recent Hamiltonian analysis of higher-curvature actions has been performed by Deruelle et al. (2009), showing the constraint algebra explicitly.)

The canonical consistency requirement of an anomaly-free, covariant theory then states that the algebra of effective constraints must remain first class. As long as this algebraic condition is satisfied, the theory is completely consistent. The system of constraints must remain first class after including quantum corrections, but the specific algebra may change. Canonically, the realm of consistent theories is larger than at the Lagrangian level: While corrections to the action may be of higher-curvature form, they all give rise to exactly the same constraint algebra. The Hamiltonian level, on the other hand, allows changes in the algebra as long as it remains first class; the theory may be consistently deformed. Thereby, changes to the quantum space-time structure can be captured. Whether or not non-trivial consistent deformations exist, and how their covariance can be interpreted, is a matter of analysis in specific quantum gravity models. We will illustrate the results of several models in loop quantum cosmology in the following section.

4.1 Cosmological perturbations

The general issues of constrained systems in gravity can easily be seen already for linear cosmological perturbations. For scalar modes, perturbing the lapse function as $N(t)(1 + \phi(x,t))$ and the scale factor as $a(t)(1 - \psi(x,t))$, and with a scalar matter source, the linearized components of Einstein's equation, in conformal time and longitudinal gauge,

read

$$
\begin{aligned}
\partial_c\left(\dot{\psi}+\mathcal{H}\phi\right) &= 4\pi G\dot{\bar{\varphi}}\partial_c\delta\varphi && (22)\\
\nabla^2\psi-3\mathcal{H}\left(\dot{\psi}+\mathcal{H}\phi\right) &= 4\pi G\left(\dot{\bar{\varphi}}\delta\dot{\varphi}-\dot{\bar{\varphi}}^2\phi+a^2V_{,\varphi}(\bar{\varphi})\delta\varphi\right) && (23)\\
\ddot{\psi}+\mathcal{H}\left(2\dot{\psi}+\dot{\phi}\right)+\left(2\dot{\mathcal{H}}+\mathcal{H}^2\right)\phi &= 4\pi G\left(\dot{\bar{\varphi}}\delta\dot{\varphi}-a^2V_{,\varphi}(\bar{\varphi})\delta\varphi\right) && (24)\\
\partial_c\partial^c(\phi-\psi) &= 0 && (25)
\end{aligned}
$$

with the conformal Hubble parameter $\mathcal{H}$. This set of equations is accompanied by the linearized Klein–Gordon equation for $\varphi=\bar{\varphi}+\delta\varphi$. The first two lines are of first order in time, and thus pose constraints for initial values. They are the linearized diffeomorphism constraint (22) and the Hamiltonian constraint (23). The next two lines are the diagonal and off-diagonal components of the spatial part of Einstein's equation, also linearized around Friedmann–Robertson–Walker. Using the background equations for $\mathcal{H}$ and the background scalar $\bar{\varphi}$, one can explicitly derive the Klein–Gordon equation from the rest. The system is thus overdetermined, but in a consistent way. Canonically, this follows from the fact that all equations result from a set of first-class constraints.

In these equations, perturbations ϕ, ψ and $\delta\varphi$ are directly used for the metric and the scalar. These quantities are subject to gauge transformations under changes of space-time coordinates, which also change the gauge. So far, the equations as written are in longitudinal gauge where the space-time metric remains diagonal under perturbations. More generally, scalar modes can be perturbed also in the off-diagonal part of the spatial metric by $\delta q_{ab}=\partial_a\partial_b E$, or in the space-time part by a perturbed shift vector $\delta N^a=\partial^a B$. Two new scalar perturbations E and B are introduced in addition to ϕ and ψ, and all four transform into each other under coordinate changes. For linear coordinate changes, however, the combinations

$$\Psi=\psi-\mathcal{H}(B-\dot{E}) \tag{26}$$

and

$$\Phi=\phi+\left(B-\dot{E}\right)^{\bullet}+\mathcal{H}(B-\dot{E}) \tag{27}$$

and a similar one for the matter perturbation remain unchanged as specified by Bardeen (1980); they form gauge-invariant observables of the linear theory. From a canonical perspective, these combinations are invariant under the flow generated by the constraints.

The ungauged evolution equations, containing E and B in addition to ϕ and ψ, can be expressed completely in terms of the gauge-invariant variables, as required on physical grounds. Here, the first-class nature of constraints is important which ensures that the constraint equations, and thus (22) and (23), are gauge invariant. Consistency as well as gauge invariance of the equations of motion is thus guaranteed by having a first-class algebra of constraints. Also here, the first-class nature serves as a complete requirement for covariance.

4.2 Gauges and frames

A first-class algebra of constraints ensures that physical evolution can be formulated fully in gauge-invariant terms. By the same property of the constrained system, also frame-independence is guaranteed: The gauge algebra corresponds to deformations of the spatial hypersurfaces given by a time function $t = \text{const}$. Gauge invariant variables are insensitive to these deformations, and thus to the choice of time. But a time function (without specifying which spatial coordinates are to be held fixed) does not uniquely tell us how to take time derivatives for the equations of motion; this is only accomplished when we also specify a time-evolution vector field t^a (such that $t^a \partial_a t = 1$). For a given time function, there are many choices for t^a, and the time-evolution vector field may be changed independently of the foliation.

A fixed foliation of space-time into spatial slizes with unit normals n^a allows us to express the freedom in choosing time-evolution vector fields by the lapse function N and the shift vector N^a (with $N^a n_a = 0$) such that $t^a = Nn^a + N^a$. These are exactly the functions which appear in the constraints generating evolution: for a fixed choice of t^a, or N and N^a, Hamiltonian equations of motion are $\dot{f} = \mathcal{L}_{t^a} f = \{f, H[N, N^a]\}$ for any phase-space function f. Since N and N^a appear as multipliers of first-class constraints, they can be chosen arbitrarily (except that we would like $N > 0$ for evolution toward the future). For a consistent set of first-class constraints, we can thus choose the frame freely, and different frame choices are guaranteed to produce consistent results. (In a space-time treatment, observables which are gauge as well as frame-independent can be derived following Ellis and Bruni (1989); Bruni et al. (1992). In a reduced phase-space treatment, where one uses observables solving the classical constraints, one is working at a gauge-invariant, but not automatically frame-independent level.)

In classical relativity, cosmological perturbation equations can often be derived in much simper ways when a space-time gauge is chosen, such as the longitudinal gauge above or the uniform one where only matter fields are perturbed. Since gauge transformations are known to correspond to space-time coordinate transformations, one can directly verify that such gauges are possible. Moreover, choosing a gauge before deriving equations of motion from an action or Hamiltonian is equivalent to choosing a gauge in the general equations of motion. The situation is thus completely unambiguous.

When quantum effects are included, either in a full quantum theory or in an effective manner, the constraints change by quantum correction terms. Equations of motion change, as expected, and so do the form of gauge-invariant expressions since it is the constraints which generate gauge transformations. In such a situation, gauge transformations and suitable gauge fixings can be analyzed only after the quantization has been performed and the corrected constraints are known. If gauge fixings are employed before quantization or before determining effective constraints, the choice of gauge fixing may not be compatible with the resulting corrected gauge transformations. Moreover, choosing different gauge fixings before doing the same kind of quantization would in general lead to different final results, making the procedure ambiguous even beyond unavoidable quantization ambiguities. Similarly, a reduced phase space quantization is based on frame-fixing, although no

gauge need be fixed.

For the different approaches used in relation to loop quantum gravity, several examples exist. Campiglia et al. (2006) develop methods to deal with a discretization possibly breaking gauge symmetries. Similar methods have then be used by Campiglia et al. (2007) in spherically symmetric models with (partial) gauge fixing. Also the hybrid quantizations of Gowdy models by Martín-Benito et al. (2008) rely on gauge fixing of the inhomogeneous generators. Bahr and Dittrich (2009b) construct discretized theories in three dimensions, respecting the space-time gauge, but argue in Bahr and Dittrich (2009a) that this may not be possible in four space-time dimensions. Laddha (2007) and Laddha and Varadarajan (2008) quantize 2-dimensional parameterized models of field theories by the Dirac procedure and represent observables on the resulting physical Hilbert space. In this treatment, the discretization does not lead to inconsistencies but possibly to deformations of classical algebras of observables. Finally, reduced phase space methods fixing the frame by referring to an extra dust field are developed by Giesel et al. (2007a,b) for cosmological perturbations, and by Giesel et al. (2009) in spherical symmetry. Other treatments for cosmological perturbations are used, e.g., by Artymowski et al. (2009) and Mielczarek (2010) in gauge-fixed versions and by Puchta (2009) in a frame-fixed (reduced phase-space) way. Rovelli and Vidotto (2008) provide a proposal by which cosmological perturbations, taking into account space-time discreteness, might be implementable by a consistent first-class algebra of constraints. So far, this has been realized for coupling two independent homogeneous patches.

The only valid treatment of a complicated gauge theory is by working without restrictions of the gauge throughout the quantization procedure, until the final gauge algebra resulting from the corrected constraints has been confirmed to be consistent. Here, the anomaly problem, confirming that a consistent deformation is realized, must be faced head-on and cannot be evaded. In the final equations one may choose one of the allowed gauges for further analysis, but gauges cannot be used to simplify the quantization. In the rest of this exposition, we follow these lines to illustrate the consistency of several effective sets of constraints incorporating some of the discreteness effects of loop quantum gravity.

5 Consistent effective discrete dynamics

If loop quantum gravity has a chance of being a viable quantum theory of gravity, the form of discrete quantum geometry it implies must give rise to effective dynamical equations satisfying the consistency conditions of a covariant theory. At the Hamiltonian level, this requires a first-class algebra of constraints. After the preparation in the preceding sections, we can now see what specific models indicate.

5.1 Constraint algebra

Consistent deformations implementing the effects of loop quantum gravity have been found in several different cases. Most of them use inverse-triad corrections, which have been in-

corporated successfully in spherically symmetric models by Bojowald and Reyes (2009); Bojowald et al. (2009c) as well as linear perturbations around spatially flat Friedmann–Robertson–Walker models by Bojowald et al. (2008, 2009b). The situation for holonomy corrections is more restrictive; here, certain versions have been realized in spherically symmetric models by Reyes (2009) as well as for linear tensor and vector modes around Friedmann–Robertson–Walker models in Bojowald and Hossain (2007, 2008). However, so far no inhomogeneous model has been found where holonomy corrections in a complete form would be consistent. Here, the requirement of anomaly-freedom appears very restrictive.

Inverse-triad corrections have been implemented consistently in several settings, and so are not ruled out by consistency considerations. In particular, corrections from the discreteness of quantum geometry are allowed and do not necessarily spoil covariance. The specific form of their implementation then tells us if and how space-time structures have to change due to quantum effects.

Based on formulas such as (14), inverse-triad corrections arise for any term in the Hamiltonian constraint bearing components of the inverse densitized triad, such as $1/\sqrt{|\det E^a_i|}$. Bojowald et al. (2008) have consistently implemented these corrections for linear inhomogeneities, where the corrected constraint algebra is of the form (21) but with

$$K^a = \mathcal{L}_{M^a} N^a - \mathcal{L}_{N^a} M^a + \bar{\alpha}^2 \bar{N} a^{-1/2} \partial^a (\delta M^0 - \delta N^0)$$

while $K^0 = \mathcal{L}_{M^a} \delta N^0 - \mathcal{L}_{N^a} \delta M^0$ retains its classical form. In addition to the contribution $\bar{N} a^{-1/2} \partial^a (\delta M^0 - \delta N^0)$, which is expected classically for a linearization around Friedmann–Robertson–Walker models with $N_i = \bar{N} + \delta N_i$, there is the function $\bar{\alpha}$ (depending on the background scale factor a) arising from inverse-triad corrections. An algebra of the same form arises for spherically symmetric models, with different matter couplings as discussed by Reyes (2009). The constraint algebra is anomaly-free: the system of constraints remains first class. But it is not exactly the classical algebra, and thus deformed. Inverse-triad corrections from loop quantum gravity cannot amount to higher-curvature corrections to the action since this would leave the constraint algebra unchanged. Rather, these corrections can only be understood as deforming local space-time symmetries while keeping covariance realized.

5.2 Cosmological perturbations

Consistent versions for quantum-corrected constraints allow one to analyze their implications for the dynamics. When constraints are corrected, not just evolution equations change but also the gauge transformations generated by the constraints. Thus, expressions for gauge-invariant observables depend differently on perturbations of the fields, which by itself may give rise to new effects. Other implications then follow from studying the dynamical evolution of gauge-invariant observables. (Some quantum-gravity corrections have been implemented in gauge-fixed versions. They are formally consistent, but quite arbitrary in their implementation. For instance, it remains unclear how different gauge-fixings,

all done before quantization, might affect the results. Moreover, some physical effects due to corrections to gauge-invariant observables can easily be overlooked.)

For linear perturbation equations around Friedmann–Robertson–Walker models, cosmological perturbation equations are the main application of consistent deformations. With a consistent deformation, perturbation equations form a closed set and can be written fully in terms of gauge-invariant variables. For inverse-triad corrections, as derived by Bojowald et al. (2009b), they take the form

$$\partial_c \left(\dot{\Psi} + \mathcal{H}(1+f)\Phi \right) = \pi G \frac{\bar{\alpha}}{\bar{\nu}} \dot{\bar{\varphi}} \partial_c \delta\varphi^{\rm GI}$$

as the corrected time-space part of Einstein's equation,

$$\begin{aligned} & \Delta(\bar{\alpha}^2 \Psi) - 3\mathcal{H}(1+f) \left(\dot{\Psi} + \mathcal{H}\Phi(1+f) \right) \\ = \;& 4\pi G \frac{\bar{\alpha}}{\bar{\nu}} (1+f_3) \left(\dot{\bar{\varphi}} \delta\dot{\varphi}^{\rm GI} - \dot{\bar{\varphi}}^2 (1+f_1)\Phi + \bar{\nu} a^2 V_{,\varphi}(\bar{\varphi}) \delta\varphi^{\rm GI} \right) \end{aligned}$$

as the corrected time-time part, and

$$\begin{aligned} & \ddot{\Psi} + \mathcal{H} \left(2\dot{\Psi} \left(1 - \frac{a}{2\bar{\alpha}} \frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}a} \right) + \dot{\Phi}(1+f) \right) + \left(2\dot{\mathcal{H}} + \mathcal{H}^2 \left(1 + \frac{a}{2} \frac{\mathrm{d}f}{\mathrm{d}a} - \frac{a}{2\bar{\alpha}} \frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}a} \right) \right) \Phi(1+f) \\ = \;& 4\pi G \frac{\bar{\alpha}}{\bar{\nu}} \left(\dot{\bar{\varphi}} \delta\dot{\varphi}^{\rm GI} - a^2 \bar{\nu} V_{,\varphi}(\bar{\varphi}) \delta\varphi^{\rm GI} \right) \end{aligned}$$

as the diagonal space-space part. All corrections f, f_1, f_3 and h below are determined from the basic inverse-triad corrections $\bar{\alpha}$ (for the gravitational part of the constraint) and $\bar{\nu}$ (for the kinetic term of the matter part). Since these are background corrections, their form can easily be computed in isotropic models, suitably parameterized for all ambiguities and lattice refinement.

The off-diagonal space-space part also implies a non-trivial equation, with an unexpected consequence: While the classical analog would simply identify $\Phi = \Psi$, the corrected equation implies $\Phi = \Psi(1+h)$ with a quantum correction by $h \neq 0$. This may be interpreted as an effective anisotropic stress contribution, but it results from a correction to quantum gravity, not to matter.

As a second implication, we may have non-conservation of power on large scales as pointed out by Bojowald et al. (2007b, 2009b).[11] This may be important for inflationary structure formation, where the long evolution times while modes are outside the Hubble radius would make even a weakly changing size of the overall power significant. Both of these effects are difficult to see in gauge-fixed treatments, such as the longitudinal or uniform gauge, or in frame-fixed versions based on reduced phase space quantizations.

[11] *Note added:* Recently it turned out that power, due to an unexpected cancellation, is conserved on large scales for the currently available equations with inverse-triad corrections, as derived by Bojowald and Calcagni (2010). So far, however, there is no general argument for the conservation of power in the presence of modified space-time structures. The analysis of Bojowald et al. (2010) has shown that interesting effects for potential observations mainly arise from corrections to the running of indices.

Since scalar cosmological perturbations have been consistently formulated only for inverse-triad corrections, no version is yet able to evolve perturbations through a bounce for which holonomy corrections are required. (Implementing holonomy corrections only for the background leads to inconsistent evolution equations for inhomogeneities.)

In addition to inverse-triad corrections, holonomies and quantum back-reaction must be implemented to obtain a full picture from loop quantum gravity. While consistent deformations are not yet known for the latter two corrections, Bojowald and Skirzewski (2008) have formulated quantum back-reaction in a cosmological setting. In general, in quantum gravity this requires the inclusion of moments between all degrees of freedom of gravity and matter, including quantum correlations between them. By setting the quantum variables of gravity as well as its quantum correlations to zero, one obtains the effective equations of quantum field theory on a curved space-time as a limit. Including leading order corrections from the gravitational quantum variables provides quantum field theory on a quantum space-time. While such limiting cases can be realized explicitly at the effective level, much still remains to be done for a detailed analysis of specific properties.

5.3 Causality

Several examples illustrate the importance of a consistent constraint algebra, rather than just any deformation as allowed in gauge-fixed or frame-fixed treatments. We have already seen that some effects in cosmological perturbation equations can be obtained only when neither gauge nor frame are fixed before the theory is quantized or effective constraints are derived. Another example is the realization of causality, as Bojowald and Hossain (2008) studied it by comparing the propagation of gravitational waves to that of light.

With quantum-gravity corrections, the gravitational as well as the Maxwell Hamiltonian change compared to the classical expressions, affecting evolution equations and their plane-wave solutions. The gravitational contribution to the Hamiltonian constraint for inverse-triad corrections is

$$H_{\rm G} = \frac{1}{16\pi G}\int_\Sigma {\rm d}^3x \alpha(E^a_i)\frac{E^c_jE^d_k}{\sqrt{|\det E|}}\left(\epsilon_i{}^{jk}F^i_{cd} - 2(1+\gamma^2)K^j_{[c}K^k_{d]}\right)$$

implying the linearized wave equation

$$\frac{1}{2}\left(\frac{1}{\alpha}\ddot{h}^i_a + 2\frac{\dot{a}}{a}\left(1-\frac{2a{\rm d}\alpha/{\rm d}a}{\alpha}\right)\dot{h}^i_a - \alpha\nabla^2 h^i_a\right) = 8\pi G\Pi^i_a$$

for the tensor mode h^i_a on a cosmological background with scale factor a and source-term Π^i_a. By a plane-wave ansatz, we derive the dispersion relation $\omega^2 = \alpha^2k^2$ for gravitational waves. Since $\alpha > 1$ for perturbative corrections, there is a danger of gravitational waves being super-luminal.

With these corrections, gravitational waves are faster than light on a classical background. For a meaningful comparison, however, we should use the speed of gravitational

waves in relation to the physical speed of light on the same background, which should receive quantum corrections, too. Here, the Hamiltonian is

$$H_{\mathrm{EM}} = \int_\Sigma \mathrm{d}^3x \left(\alpha_{\mathrm{EM}}(q_{cd}) \frac{2\pi}{\sqrt{\det q}} E^a E^b q_{ab} + \beta_{\mathrm{EM}}(q_{cd}) \frac{\sqrt{\det q}}{16\pi} F_{ab} F_{cd} q^{ac} q^{bd} \right)$$

with two correction functions α_{EM} and β_{EM} kinematically independent of the gravitational correction α. From the wave equation

$$\partial_t \left(\alpha_{\mathrm{EM}}^{-1} \partial_t A_a \right) - \beta_{\mathrm{EM}} \nabla^2 A_a = 0$$

we obtain the dispersion relation $\omega^2 = \alpha_{\mathrm{EM}}\beta_{\mathrm{EM}}k^2$, which also is "super-luminal" compared to the classical speed of light.

Working out the requirements for anomaly-freedom with these two contributions to the Hamiltonian constraint, as done by Bojowald and Hossain (2008), we find $\alpha^2 = \alpha_{\mathrm{EM}}\beta_{\mathrm{EM}}$ and the dispersion relations are equal. Physically, for comparisons of speeds on the same background, there is no super-luminal propagation. In a gauge-fixed or frame-fixed treatment, by comparison, one could have chosen the correction functions independently of another since tensor modes of the gravitational field and the electric field make up independent physical observables. A gauge- or frame-fixed treatment could easily produce corrected equations violating causality, but this is ruled out by a complete treatment.

6 Outlook: Future dynamics

To probe a quantum theory of gravity or even arrive at predictions one must evaluate its dynamics in detail. For low-energy effects, leading corrections to classical equations must be derived. The best tool for systematic investigations in such cases is that of effective descriptions, providing the evolution of expectation values of observables in a physical state. At the same time, they can tell us much of the entire behavior of physical states.

Hamiltonian effective descriptions can be applied directly to canonical quantum gravity and cosmological models. In particular, typical implications such as the discreteness of spatial or space-time structures can then be probed, or first ensured to be consistent at all. Several examples in loop quantum cosmology have demonstrated that discreteness corrections can indeed be implemented consistently, leaving the theory covariant but deforming its local space-time symmetries. Future work must ensure that this is indeed possible for the full theory of loop quantum gravity and its effective constraints.

While isotropic solvable models of loop quantum cosmology suggest a role of bouncing cosmologies for potential scenarios, no consistent set of equations to evolve inhomogeneities through a bounce has been found. The only available options so far make use of gauge (or frame) fixings before quantization, and thus miss crucial aspects of space-time structures. Any mismatch of growing modes in the collapse and expansion phases can easily be enhanced by cosmic evolution, providing opportunities for potential observations but also requiring extreme care in finding fully consistent equations. Inhomogeneous cosmological scenarios remain uncertain, and with it follow-up issues such as the entropy problem.

What models investigated so far suggest, in many different versions, is that the classical algebra of space-time diffeomorphisms is deformed but not violated. It is then clear that quantum corrections cannot merely amount to higher-curvature terms in an effective action, although such terms may appear, too. Instead, quantum structures of space-time must change by quantum effects. While space-time covariance is no longer realized in the standard sense, from the Hamiltonian perspective the effective theories remain completely consistent and covariant with an underlying first-class algebra of gauge generators. As indicated by the initial quote from Dirac (1958), this sense of covariance, from a general perspective, is the appropriate one. Its consistent implementation, without fixing gauge or frame, can tell us a great deal about the fundamental structures of space and time.

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